MODERN PHYSICS (Paper: PHS-A-CC-3-7-TH Credit:4)

1. Radiation and its nature (15 Lectures)

(a) Blackbody Radiation, Planck's quantum hypothesis, Planck's constant (derivation of Planck formula is not required). Photoelectric effect and Compton scattering – light as a collection of photons. Davisson-Germer experiment. De Broglie wavelength and matter waves. Wave-particle duality. Wave description of particles by wave packets. Group and Phase velocities and relation between them. Probability interpretation: Normalized wave functions as probability amplitudes.

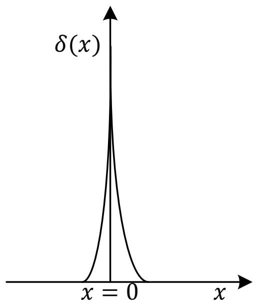
(b) Two-slit experiment with photons and electrons. Linear superposition principle as a consequence.

(c) Position measurement, gamma ray microscope thought experiment. Heisenberg uncertainty principle (Statement with illustrations). Impossibility of a particle following a trajectory.

Mathematical tools

1. **Dirac Delta function**

Discrete sources such as point charges, elementary currents, current shells, etc constitute singularities of electromagnetic fields. One convenient method treat these discrete sources is the use of Dirac delta function. In one dimension Dirac delta function at a point $x = x_0$ is designated by $\delta(x - x_0)$ and at $x = 0$ by $\delta(x)$. It has the following properties. 1. $\delta(x - x_0) = \begin{cases} 0, & for\ x \neq x_0 \\ \infty & for\ x = x_0 \end{cases}$

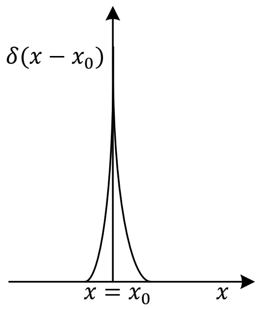


2. $\int_a^b \delta(x - x_0)dx = \begin{cases} 1, & if\ x_0\ is\ in\ (a,b) \\ 0 & if\ x_0\ is\ not\ in\ (a,b) \end{cases}$

3. $\int_a^b f(x)\delta(x - x_0)dx = \begin{cases} f(x_0), & if\ x_0\ is\ in\ (a,b) \\ 0 & if\ x_0\ is\ not\ in\ (a,b) \end{cases}$

4. $\int_a^b f(x)\delta'(x - x_0)dx = \begin{cases} -f'(x_0), & if\ x_0\ is\ in\ (a,b) \\ 0 & if\ x_0\ is\ not\ in\ (a,b) \end{cases}$

5. $\delta(\vec{r} - \vec{r}_0) = \begin{cases} 0, & for\ \vec{r} \neq \vec{r}_0 \\ \infty & for\ \vec{r} = \vec{r}_0 \end{cases}$



6. $\iiint_V \delta(\vec{r} - \vec{r}_0)dV = \begin{cases} 1, & if\ \vec{r}_0\ is\ in\ V \\ 0 & if\ \vec{r}_0\ is\ not\ in\ V \end{cases}$

7. $\int_V f(\vec{r})\delta(\vec{r} - \vec{r}_0)dV = \begin{cases} f(\vec{r}_0), & if\ \vec{r}_0\ is\ in\ V \\ 0 & if\ \vec{r}_0\ is\ not\ in\ V \end{cases}$

Note that the Dirac delta function is not continuous one, it can be considered as the limit of an ordinary function. For example $\boxed{\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi}\epsilon}e^{-\frac{x^2}{\epsilon^2}}}$

More properties of delta function. $\boxed{\delta(x) = -\delta(-x); \quad \delta'^{(x)} = -\delta'^{(-x)}; \quad \delta(ax) = \frac{\delta(x)}{a}}$

Another important property of Dirac delta function comes from its relation to the divergence or a Laplacian operator

$$\vec{\nabla}.\frac{\vec{r}}{r^3} = \vec{\nabla}.\frac{\vec{r}}{r^3} = \left[\frac{1}{r^3}\vec{\nabla}.\vec{r} + \vec{r}.\vec{\nabla}\left(\frac{1}{r^3}\right)\right] = \left[\frac{3}{r^3} + \vec{r}.(-3r^5\vec{r})\right] = \left[\frac{3}{r^3} - \frac{3}{r^3}\right] = 0 \quad \text{for } r \neq 0$$

And $\vec{\nabla}.\frac{\vec{r}}{r^3}$ becomes indeterminate as $r \to 0$. Hence to avoid this situation we can represent $\vec{\nabla}.\frac{\vec{r}}{r^3}$ by Dirac Delta function. Applying divergence theorem to a small sphere of radius R around the origin we get

$$\iiint_V \vec{\nabla}.\frac{\vec{r}}{r^3}\, dV = \oiint_S \frac{\vec{r}.\hat{n}dS}{r^3} = \frac{1}{R^2}\oiint_S dS = \frac{1}{R^2}4\pi R^2 = 4\pi = 4\pi\iiint_V \delta(\vec{r})dV = \iiint_V 4\pi\delta(\vec{r})dV$$

since the result is true regardless to how small the radius, one can replace $\vec{\nabla}.\frac{\vec{r}}{r^3}$ by $4\pi\delta(\vec{r})$.

Hence we can write $\boldsymbol{\delta(\vec{r}) = \frac{1}{4\pi}\vec{\nabla}.\frac{\vec{r}}{r^3} = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right)}$

2. **Binomial expansion** $(1+x)^n = 1 + {}^n_1c\, x + {}^n_2c\, x^2 + \cdots \ldots \ldots = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^2 + \ldots\ldots$

Note that for n is a positive integer the series terminate for all x. but when n is not a positive integer, the series does not terminate. The infinite series is convergent for $|x| < 1$

$$\boldsymbol{(a+b)^n = a^n + {}^n_1c\, a^{n-1}b^1 + {}^n_2c\, a^{n-2}b^2 + \cdots = 1 + na^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots}$$

3. **Taylor and McLaurin series:** A **Taylor series** is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function $f(x)$ about a point $x = a$ is given by

$$f(x) = f(a) + \frac{1}{1!}(x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(x-a) + \cdots \ldots + \frac{1}{n!}(x-a)^n f^{(n)}(x-a)$$

If $a = 0$, the expansion is known as a **Maclaurin series.**

4. Power series with real variables

Powers of natural numbers	
$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$	$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\ldots\ldots = \ln 2$
$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$	$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\ldots = \frac{\pi}{4}$
$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$	$\sum_{k=1}^{\infty}\frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\ldots = \frac{\pi^2}{6}$

Special power series	
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (for all values of x)	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\ldots$ (for $-1 < x < 1$)
$a^x = 1 + x\ln a + \frac{(x\ln a)^2}{2!}\ldots$ for all values of x	$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}\ldots\ldots$ (for $-\frac{\pi}{2} < x < \frac{\pi}{2}$)
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\ldots$ (for all values of x)	$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315}\ldots\ldots$ (for $-\frac{\pi}{2} < x < \frac{\pi}{2}$)
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\ldots$ (for all values of x)	$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5}\ldots\ldots\ldots\ldots$ (for $-1 < x < 1$)
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}\ldots$ (for all values of x)	$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x\ldots\ldots\ldots\ldots$ (for $-1 < x < 1$)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$ (for all values of x)	$\tan^{-1} x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}\ldots\ldots$ (for $-1 < x < 1$)

5. Vector differential operators in different co-ordinate systems

Cylindrical co-ordinates(ρ,φ,z)	Spherical co-ordinate (r,θ,φ)	Remarks
$\vec{\nabla}\phi=\hat{\rho}\frac{\partial\phi}{\partial\rho}+\hat{\varphi}\frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}+\hat{k}\frac{\partial\phi}{\partial z}$ $\vec{\nabla}.\vec{A}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial}{\partial\varphi}(A_\varphi)+\frac{\partial}{\partial z}(A_z)$ $\vec{\nabla}\times\vec{A}=\frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\varphi} & \hat{k}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\varphi & A_z\end{vmatrix}$ $\nabla^2\varphi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}(\phi)+\frac{\partial^2}{\partial\varphi^2}(\phi)$ **Elementary area** $\rho d\varphi dz$ **(curved surface)** **Elementary volume** $\rho d\rho d\varphi dz$	$\vec{\nabla}\phi=\hat{r}\frac{\partial\phi}{\partial r}+\hat{\varphi}\frac{1}{r}\frac{\partial\phi}{\partial\theta}+\hat{k}\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}$ $\vec{\nabla}.\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,A_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(A_\varphi)$ $\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ A_r & rA_\theta & r\sin\theta\,A_\varphi\end{vmatrix}$ $\nabla^2\varphi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial\phi}{\partial\varphi}\right)$ **Elementary area** $r^2\sin\theta d\theta d\varphi$ **Elementary volume** $r^2\sin\theta\, drd\theta d\varphi$	h_1 h_2 h_3 1 1 1 1 ρ 1 1 r $r\sin\theta$ $\vec{\nabla}\phi=\hat{e}_1\frac{1}{h_1}\frac{\partial\phi}{\partial u_1}+\cdots$ $\vec{\nabla}.\vec{A}=\frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3A_1)+\cdots\right\}$ $\nabla^2\varphi=\frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\cdots\right\}$

6. Integration

1. $\int\ln x\ dx=x(\ln x-1)+c$	8. $\int\frac{x}{(x^2\pm a^2)^n}dx=\frac{-1}{2(n-1)}\frac{1}{(x^2\pm a^2)^{n-1}}+c$ for $n\neq 1$
2. $\int x\,e^{ax}\ dx=e^{ax}\left(\frac{x}{a}-\frac{1}{a^2}\right)+c$	9. $\int\frac{1}{\sqrt{x^2\pm a^2}}dx=\ln(x+\sqrt{x^2\pm a^2})+c$
3. $\int x\ \ln x\ dx=\frac{x^2}{2}\left(\ln x-\frac{1}{2}\right)+c$	10. $\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}(\frac{x}{a})+c$
4. $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}(\frac{x}{a})+c$	11. $\int\frac{x}{\sqrt{x^2\pm a^2}}dx=\sqrt{x^2\pm a^2}+c$
5. $\int\frac{1}{x^2-a^2}dx=-\frac{1}{a}\coth^{-1}(\frac{x}{a})+c=\frac{1}{2a}\ln\frac{x-a}{x+a}+c$ for $x^2>a^2$	**12.** $\int\sqrt{a^2-x^2}dx=\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}(\frac{x}{a})\right]+c$
6. $\int\frac{1}{a^2-x^2}dx=\frac{1}{a}\tanh^{-1}(\frac{x}{a})+c=\frac{1}{2a}\ln\frac{a+x}{a-x}+c$ for $x^2<a^2$	13. $\int_0^\infty\cos x^2\,dx=\int_0^\infty\sin x^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}$
7. $\int\frac{x}{x^2\pm a^2}dx=\frac{1}{2}\ln(x^2\pm a^2)+c$	**14.** $\int_{-\infty}^{\infty}e^{(-\frac{x^2}{2\sigma^2})}dx=\sigma\sqrt{2\pi}$

15. $\int_{-\infty}^{\infty}x^ne^{(-\frac{x^2}{2\sigma^2})}dx=\begin{cases}1\times3\times5\times\ldots(n-1)\sigma^{n+1}\sqrt{2\pi} & \text{for } n\geq 2 \text{ and even}\\ 0 & \text{for } n\geq 1 \text{ and odd}\end{cases}$

16. $\int\sin mx\ \sin nx\ dx=\frac{\sin(m-n)x}{2(m-n)}-\frac{\sin(m+n)x}{2(m+n)}+c$ $\qquad m^2\neq n^2$

17. $\int\cos mx\ \cos nx\ dx=\frac{\sin(m-n)x}{2(m-n)}+\frac{\sin(m+n)x}{2(m+n)}+c$ $\qquad m^2\neq n^2$

18. $\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\ d\theta=\frac{m-1}{m+n}\int_0^{\frac{\pi}{2}}\sin^{m-2}\theta\cos^n\theta\ d\theta=\frac{n-1}{m+n}\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^{n-2}\theta\ d\theta$

Hence it can be reduced to one of the following $\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\ d\theta=\frac{1}{2},\int_0^{\frac{\pi}{2}}\sin\theta\ d\theta=1,\int_0^{\frac{\pi}{2}}\cos\theta\ d\theta=1,\int_0^{\frac{\pi}{2}}d\theta=\frac{\pi}{2},$

19. If $I_n=\int_0^\infty x^ne^{(-\alpha x^2)}dx$ **then** $I_n=\frac{n-1}{2\alpha}I_{n-2}$ **with** $I_0=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}},I_1=\frac{1}{2\alpha}$

20. $\int f(x)g(x)dx=f(x)\int g(x)dx-f'(x)\int(\int g(x)dx)\,dx+f''^{(x)}\int\{\int(\int g(x)dx)\,dx\}\,dx-\cdots..$

7. **Fourier series**

Any single-valued periodic function whatever can be expressed as a summation of simple harmonic terms having frequencies which are multiples of that of the given function. Let $f(x)$ be defined in the interval $(-\pi,\pi)$ and assume that $f(x)$ has the period 2π, i.e. $f(x+2\pi)=f(x)$. The Fourier series or Fourier expansion corresponding to $f(x)$ is defined as $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_0\cos nx+b_n\sin nx)$..........(1)

where the Fourier coefficient a_n and b_n are

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\ \mathrm{d}x$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\ \mathrm{d}x$$

where $n=0,1,2,\ldots$

8. **Complex form of Fourier series**

Assuming that the Series (1) converges at $f(x)$, $f(x)=\sum_{n=-\infty}^{\infty}C_n e^{in\pi x/L}$

$$\text{With } C_n=\frac{1}{L}\int_C^{C+2L}f(x)e^{-in\pi x/L}\ \mathrm{d}x=\begin{cases}\frac{1}{2}(a_n-ib_n) & n>0\\ \frac{1}{2}(a_{-n}+ib_{-n}) & n<0\\ \frac{1}{2}a_o & n=0\end{cases}$$

9. **Fourier transforms** The Fourier transform of $f(x)$ is defined as $\Im(f(x))=F(\alpha)=\int_{-\infty}^{\infty}f(x)e^{i\alpha x}\ \mathrm{d}x$

and the inverse Fourier transform of $F(\alpha)$ is $\Im^{-1}(f(\alpha))=F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\alpha)e^{i\alpha x}d\alpha$

$f(x)$ and $F(\alpha)$ are known as Fourier Transform pairs. Some selected pairs are given in

$f(x)$	$F(\alpha)$	$f(x)$	$F(\alpha)$
$1/x$	$\frac{\pi}{2}$	e^{-ax}	$\frac{a}{\alpha^2+a^2}$
$\frac{1}{x^2+a^2}$	$\frac{\pi}{a}e^{-a\alpha}$	e^{-ax^2}	$\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{\alpha^2}{4a}}$
$\frac{x}{x^2+a^2}$	$\frac{i\pi\alpha}{a}e^{-a\alpha}$	xe^{-ax^2}	$\frac{1}{4}\sqrt{\frac{\pi}{a^3}}\alpha e^{-\frac{\alpha^2}{4a}}$

10. **Gamma and beta functions**

The gamma function $\Gamma(n)$ is defined by

$$\Gamma(n)=\int_0^{\infty}e^{-x}x^{n-1}\ \mathrm{d}x\ (\mathrm{Re}\,n>0)$$
$$\Gamma(n+1)=n\Gamma(n)$$

If n is a positive integer

$$\Gamma(n+1)=n!$$
$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi};\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2};\Gamma\left(\frac{5}{2}\right)=\frac{3}{4}\sqrt{\pi}$$
$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1.3.5\ldots(2n-1)\sqrt{\pi}}{2^n}\ (n=1,2,3,\ldots)$$
$$\Gamma\left(-n+\frac{1}{2}\right)=\frac{(-1)^n2^n\sqrt{\pi}}{1.3.5\ldots(2n-1)}\ (n=1,2,3,\ldots)$$
$$\Gamma(n+1)=n!\cong\sqrt{2\pi n}n^ne^n\ (\text{Stirling's formula})$$
$$n\to\infty$$

Beta function $B(m,n)$ is defined as

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$
$$B(m,n) = B(n,m)$$
$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$$
$$B(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

11. Special functions, properties and differential equations

a) Hermite functions: Differential equation: $y'' - 2xy' + 2ny = 0$

when $n = 0,1,2,\dots$ then we get Hermite's polynomials $H_n(x)$ of degree n, given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad \text{(Rodrigue's formula)}$$

First few Hermite's polynomials are:

$$H_o(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x, H_4(x) = 16x^4 - 48x^2 + 12$$

Generating function:

$$e^{2ux-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}$$

Recurrence formulas:

$$H_n'(x) = 2nH_{n-1}(x)$$
$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Orthonormal properties: $\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = \begin{cases} 0 & \text{for } m \neq n \\ 2^n n! \sqrt{\pi} & \text{for } m = n \end{cases}$

b) Legendre's equation $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$

Solution of which is Legendre's polynomial $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$$P_0(x) = 1,\ P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{1}{2}(5x^3-3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Generating function: $\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$

Recurrence formulas: $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$ & $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$

Orthonormal properties: $\int_{-1}^{1} P_m(x)P_n(x)dx = 0\ m \neq n$ & $\int_{-1}^{1} \{P_n(x)\}^2\, dx = \frac{2}{2n+1}$ for $m = n$

Other properties: $P_n(1) = 1, P_n(-1) = (-1)^n, P_n(-x) = (-1)^n P_n(x)$

c) ***Associated Legendre functions:***

Differential equation: $(1-x^2)y'' - 2xy' + \left\{l(l+1) - \frac{m^2}{1-x^2}\right\}y = 0$

and $P_l^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x)$

where $P_l(x)$ are the Legendre polynomials stated previously, l being the positive integer.

$$P_l^n(x) = P_l(x)$$
$$\text{and } P_l^m(x) = 0 \text{ if } m > n$$

Orthonormal properties: $\int_{-1}^{1} P_n^m(x)P_l^m(x)\mathrm{d}x = 0 \; n \neq l$ & $\int_{-1}^{1}\{P_l^m(x)\}^2\,\mathrm{d}x = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$

d) ***Bessel functions: ($J_n(x)$)***

Differential equation of order n is $x^2y'' + xy' + (x^2 - n^2)y = 0 \; n \geq 0$

Expansion formula: $J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k-n}}{k!\Gamma(k+1-n)}$

Properties:

$$J_{-n}(x) = (-1)^n J_n(x) \; n = 0,1,2,\ldots$$
$$J_o'(x) = -J_1(x)$$
$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Generating function: $e^{x(s-1/s)/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$

EXPLANATION OF SPECTRAL DISTRIBUTION IN A BLACKBODY RADIATION.

Characteristics of black body radiation.

1. This radiation depends on temperature of blackbody and is independent of nature of the material of the walls.
2. A plot of energy density of radiation at wavelength λ of a black body (u_λ) against λ, at a particular temperature T shows that -

a. Energy of **emitted radiation is not confined in a single wavelength** but is **distributed amongst different wavelength 0 to ∞ continuously.**

b. The energy **is not uniformly distributed** in the radiation. u_λ is small for low value of λ. u_λ Increase with λ becomes maximum at $\lambda = \lambda_m$. u_λ decreases with further increase of λ.

c. An increase in temp. causes a decrease in λ_m such that, $\lambda_m T$= constant = 0.2896 cm-K. This relation is known as Wien's displacement law.

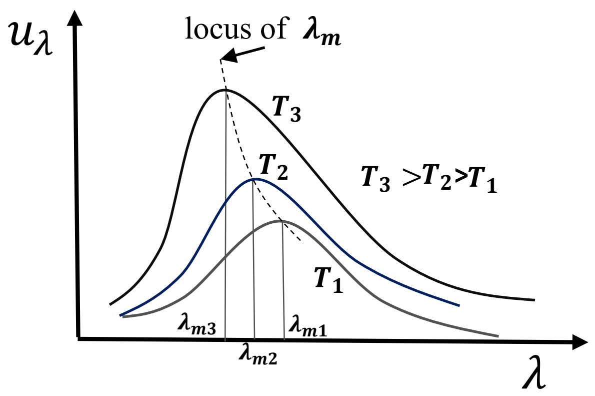


figure 1

d. An increase in temperature causes an increase in energy emission for all wavelengths.

e. The total area under each curve represents the total energy emitted by a blackbody at a particular temp. for the range of wavelength considered. This area increase with temperature and found to directly proportional to fourth power of absolute temp. i.e., U $\propto T^4$. Which is Stefan's law.

The ultra violet catastrophe-

Why does the blackbody spectrum have the shape shown in figure 1? This problem was explained at the end of 19th century by Lord Rayleigh and James Jeans. They started by considering

1) The **radiating system as composed of a collection of L.H.O.** If the oscillators **carry electric charge** they will radiate electromagnetic waves. They can also absorb electromagnetic radiation.

(NOTE THAT- According to Kirchhoff theoretically any uniformly heated enclosure or, chamber or, cavity behaves like a blackbody)

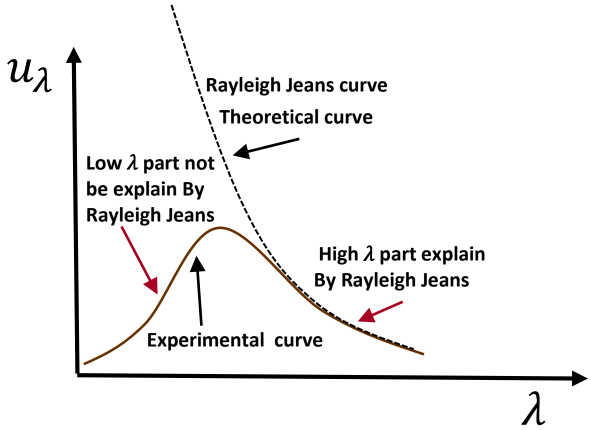
2) The atomic oscillators in the walls enclosing a cavity will continuously exchange energy with the E.M radiation with λ extending from 0 to ∞, enclose in the cavity. Ultimately equilibrium condition will be established when the energy density of E.M radiation will assume equilibrium value determine by the temperature of the cavity walls.

The radiation inside the cavity (of absolute temperature T & whose walls are perfect reflectors of E.M waves) are reflected from the walls of the cavity. The incident and reflected waves interfere to form stationary waves. **(This is three dimensional extension of standing wave in stretch string.)**
(NOTE THAT - The condition for standing waves in such a cavity is the path length of wall to wall, whatever be the direction must be a whole number of half wavelength **so that node occurs at each reflecting surface**)

Theorem gives the number of independent standing waves in the wavelength interval λ & λ +dλ per unit volume turned out to be $\boldsymbol{G(\lambda)d\lambda = \frac{8\pi}{\lambda^4}d\lambda}$

3) **The classical law of equipartition of energy** was employed to find the energy of each independent vibration. According to which a L.H.O has two degrees of freedom then **average energy** $\bar{\epsilon} = \boldsymbol{kT}$

$(k = boltzmann\ constant)$



Based on these assumption Lord Rayleigh and James Jeans obtain the formula of energy density of radiation between λ & λ +dλ = u_λ dλ = $G(\lambda)d\lambda\, \boldsymbol{kT} = \frac{8\pi \boldsymbol{kT}}{\lambda^4}d\lambda$.

So the total energy density at T$\neq$ 0 as predicted by Rayleigh and Jeans law –

$U = \int_0^\infty u_\lambda\, d\lambda = \int_0^\infty \frac{8\pi \boldsymbol{kT}}{\lambda^4}d\lambda = \frac{8\pi kT}{3}\left[\frac{1}{0} - \frac{1}{\infty}\right]$ $\quad\therefore$ U $\rightarrow\ \infty$

Hence Rayleigh and Jeans formula predicts that for low λ (i.e. at the ultraviolet end of spectrum) U$\rightarrow\ \infty$ **figure -2 (which cause a disaster or catastrophe).**

This phenomenon U$\rightarrow\ \infty$ at $\lambda \rightarrow 0, T \neq 0$ is known as ultra violet catastrophe.

NOTE THAT-this formula is found to **agree with experiment only for high value of λ.**

Plank's hypothesis.

Plank realised that physics of micro world are was different and needed an approach witch was not classical. Plank made the following assumption-

1) The radiating system is considered to be built out of numerous electric dipoles which are L.H.O of microscopic dimension. Radiation inside cavity is due to oscillation of these electric dipoles.
2) The oscillator due to oscillation emits E.M. radiation into blackbody chamber and absorb them as well at a particular temperature oscillator maintain equilibrium with radiation within blackbody chamber.
3) Oscillator's **energy is quantised** 'zero' or 'integral multiple of $\boldsymbol{h\nu}$'. The oscillator cannot have any arbitrary value of energy. They have discontinuous value of energy.

4) **Energy exchange occurs in a quantised manner in units of $h\nu$** between oscillator and radiation. (The last two assumptions are non-classical.)

[**Note that:** On the basis of above assumption, the average energy of an oscillator

$$< E >= \frac{\sum_{n=0}^{\infty} N_n E_n}{\sum_{n=0}^{\infty} N_n} = \frac{\sum_{n=0}^{\infty} N_0 e^{-nh\nu/kT} nh\nu}{\sum_{n=0}^{\infty} N_0 e^{-nh\nu/kT}}$$

Let us take $e^{-h\nu/kT} = x$, therefore, $e^{-nh\nu/kT} = x^n$ Hence, $< E >= \frac{h\nu\sum_{n=0}^{\infty} nx^n}{\sum_{\eta=0}^{\circ} x^n}$

$$< E >= \frac{h\nu[0 + x + 2x^2 + 3x^3 + \cdots \ldots \ldots]}{[1 + x + x^2 + x^3 + \cdots \ldots \ldots]} = \frac{h\nu x[1 + 2x + 3x^2 + \cdots \ldots \ldots]}{[1 + x + x^2 + x^3 + \cdots \ldots \ldots]}$$

$$= \frac{h\nu x(1-x)^{-2}}{(1-x)^{-1}} = h\nu x(1 - x)^{-1} = \frac{h\nu}{1-x} = \frac{h\nu}{x^{-1}-1} = \frac{h\nu}{e^{h\nu/kT}-1}$$

Theorem gives the number of independent standing waves in the wavelength interval λ & λ +dλ per unit volume turned out to be $G(\lambda)d\lambda = \frac{8\pi}{\lambda^4} d\lambda$.

Average energy of (Plank) oscillator = $\overline{\in} = \frac{h\nu}{e^{h\nu/kT}-1} = \frac{hc}{\lambda\,(e^{hc/\lambda kT}-1)}$

Then Plank's law of radiation gives the formula $u_\lambda\, d\lambda = G(\lambda)d\lambda \frac{hc}{\lambda\,(e^{\frac{hc}{\lambda kT}}-1)} = \frac{8\pi}{\lambda^4} \frac{hc\, d\lambda}{\lambda\,(e^{\frac{hc}{\lambda kT}}-1)}$

OR, $$\boxed{u_\lambda\, d\lambda = \frac{8\pi}{\lambda^5}\frac{hc\, d\lambda}{(e^{hc/\lambda kT}-1)}}$$

Deduction from Plank's law-

A. **Wien's distribution law-** If $\lambda \to 0$ then $\frac{hc}{\lambda kT} \to \infty$

then $(e^{\frac{hc}{\lambda kT}} - 1) \to e^{\frac{hc}{\lambda kT}}$ **Then** $u_\lambda\, d\lambda = \frac{8\pi}{\lambda^5}\frac{hc\, d\lambda}{e^{hc/\lambda kT}} = \frac{A}{\lambda^5} f(\lambda T)\, d\lambda$

B. **Wien's displacement law-**

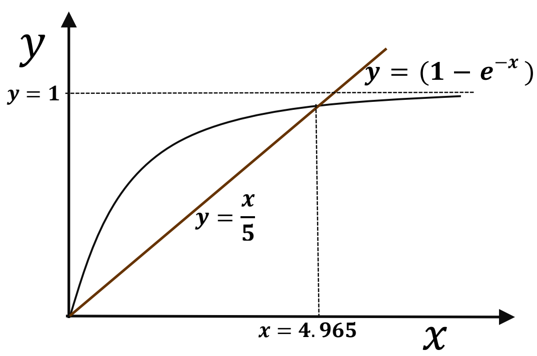
As $u_\lambda\, d\lambda = \frac{8\pi}{\lambda^5}\frac{hc\, d\lambda}{(e^{hc/\lambda kT}-1)}$

Then $u_\lambda\, d\lambda$ will be maximum when $\lambda^5(e^{hc/\lambda kT} - 1) = Z(say)$ **is minimum.** $i.e, \frac{dZ}{d\lambda} = 0$

$$\therefore \lambda^5\, e^{\frac{hc}{\lambda kT}} \left(-\frac{1}{\lambda^2}\right)\left(\frac{hc}{kT}\right) + 5\lambda^4(e^{hc/\lambda kT} - 1) = 0$$

Or, $5\lambda^4(e^{hc/\lambda kT} - 1) = \lambda^3\, e^{\frac{hc}{\lambda kT}}\left(\frac{hc}{kT}\right)$

Dividing both side by $5\lambda^4\, e^{\frac{hc}{\lambda kT}}$,



we get $\left(\mathbf{1}-\boldsymbol{e}^{-hc/\lambda kT}\right)=\frac{\mathbf{1}}{\mathbf{5}}\left(\frac{hc}{\lambda kT}\right)$ Let assume $x=\left(\frac{hc}{\lambda kT}\right)$ then$\left(\mathbf{1}-\boldsymbol{e}^{-x}\right)=\frac{\mathbf{1}}{\mathbf{5}}\boldsymbol{x}$,

which is a transcendental equation and has a solution $x = 4.965$

Then $\frac{hc}{\lambda_m kT}=4.965$ $\quad\therefore \lambda_m T=\frac{hc}{4.965k}=0.29$ cm-K $\quad\therefore \boldsymbol{\lambda_m T = CONSTANT}$

C. **Rayleigh and Jeans formula-**

If $\lambda\to\infty$ then $\left(\boldsymbol{e}^{\frac{hc}{\lambda kT}}-\mathbf{1}\right)\to\mathbf{1}+\frac{\boldsymbol{hc}}{\boldsymbol{\lambda kT}}-\mathbf{1}=\frac{\boldsymbol{hc}}{\boldsymbol{\lambda kT}}$

λ	0	∞
x	∞	0

Then $u_\lambda\, d\lambda=\frac{8\pi}{\lambda^5}\frac{hc\,d\lambda}{\left(e^{hc/\lambda kT}-1\right)}\approx\frac{8\pi}{\lambda^5}\frac{hc\,d\lambda}{\frac{hc}{\lambda kT}}=\frac{8\pi kT}{\lambda^4}d\lambda.$

D. **Stefan's law-**

So the total energy density at T$\neq 0$ as predicted by Plank's law

$$U=\int_0^\infty u_\lambda\, d\lambda=\int_0^\infty\frac{8\pi}{\lambda^5}\frac{hc\,d\lambda}{\left(e^{hc/\lambda kT}-1\right)}$$

Let $x=\left(\frac{hc}{\lambda kT}\right)$ then $d\lambda=-\frac{hc}{x^2kT}dx$

$$\therefore U=\int_\infty^0\frac{8\pi}{\left(\frac{hc}{xkT}\right)^5}\frac{hc}{\left(e^x-1\right)}\left(-\frac{hc}{x^2kT}\right)dx=\frac{8\pi k^4T^4}{h^3c^3}\int_0^\infty\frac{x^3}{\left(e^x-1\right)}dx=\frac{8\pi k^4T^4}{h^3c^3}\left(\frac{\pi^4}{15}\right)$$

$$\therefore \boldsymbol{U=\sigma T^4}$$

NOTE THAT: The energy density u_λ is simply related to intensity of emitted radiation E_λ as $\boxed{\boldsymbol{E_\lambda=\frac{C}{4}u_\lambda}}$

Criticism of Planck's hypothesis

1. Planck assumed that the atomic oscillators had energy $E_n=nh\nu, n=0,1,2,3,\ldots,\infty$(see Fig. 2.6). However, later developments showed that this value is not correct. The correct value of energy of a quantum linear harmonic oscillator is given by $E_n=\left(n+\frac{1}{2}\right)h\nu, n=0,1,2,3,\ldots,\infty$

This error, however does not make any difference to Planck's conclusions regarding blackbody spectrum.

2. Planck treated energy of oscillators in the cavity walls as quantized, but radiation within cavity as a continuous wave. However, interaction between matter (oscillator) and radiation occurred in a quantized (discontinuous) manner. Again the classical Maxwell-Boltzmann distribution law was used in the treatment. It follows that Planck's treatment was based on certain arbitrary assumption and had certain contradictions as it blended classical ideas with novel non classical ideas.

P1: (i) What is the average energy of a one-dimensional classical oscillator?

(ii) Write down the average energy of a Planck oscillator and show that it leads to the classical result under the condition of high temperature.

⇨ (i) Average (or mean) energy of a classical oscillator using equipartition of energy theorem.

Consider a one-dimensional classical oscillator of mass m, position coordinate x, momentum p, stiffness constant k_0. Energy of oscillator is $E =$ kinetic energy $+$ potential energy $= \frac{p^2}{2m} + \frac{1}{2}k_0x^2$

It has 2 quadratic terms: one in x, another in p. Again x and p are 2 independen variables.

So according to equipartition of energy theorem each quadratic term will contribute $\frac{1}{2}kT$

(Where $k =$ Boltzmann constant) So the mean energy is $\langle E\rangle = \frac{1}{2}kT + \frac{1}{2}kT$ or, $\langle E\rangle = kT$

(ii) The average energy of a Planck oscillator is $\langle E\rangle = \frac{h\nu}{e^{\frac{h\nu}{kT}}-1} \equiv \frac{h\nu}{e^{\beta h\nu}-1}$

where $h =$ Planck's constant; $k =$ Boltzmann constant; $\nu =$ linear frequency of oscillator; $T =$ absolute temperature of oscillator and $\beta = \frac{1}{KT}$.

Under the condition of high temperature: $kT =$ large and so $h\nu \ll kT \Rightarrow \frac{h\nu}{kT} = \beta h\nu \ll 1$. As $\beta h\nu$ is small, $e^{\beta h\nu}$ can be expanded AS $e^{\beta h\nu} = 1 + \beta h\nu + \frac{1}{2!}(\beta h\nu)^2 + \cdots \simeq 1 + \beta h\nu$

$$\therefore \langle E\rangle = \frac{h\nu}{1 + \beta h\nu - 1} = \frac{1}{\beta} \Rightarrow \langle E\rangle = kT$$

So classical result is obtained under high temperature limit.

P2: Find the number of photons emitted per second by a* $\mathbf{40\ W}$ *source of monochromatic light of wavelength* $\mathbf{6000\AA}$*.

⇨ Let the number of photons be n. Then $nh\nu = E$

Then $n = \frac{E}{h\nu} = \frac{E\lambda}{hc} = \frac{40\times6000\times10^{-10}}{6.63\times10^{-34}\times3\times10^{8}} = 12.06\times10^{19}$

P3: $\mathbf{60\ kW}$ ***broadcasting antenna is emitting electromagnetic waves at a frequency of*** $\mathbf{4MHz}$***. Explain whether or not will you treat the emitted radiation quantum mechanically?***

⇨ The energy emitted by the antenna per second is $E = 60\times1000 = 60{,}000$ J

If n be the number of photons being emitted per second, we get $E = nh\nu$

Thus, we have $n = \frac{E}{h\nu}$. Substituting the values of E, h and ν, we get $n = \frac{60{,}000}{6.63\times10^{-34}\times4\times10^{6}} = 2.2\times10^{31}$

Since n is exceedingly large, the quantum nature of the emitted radiation is not meaningful.

P4: (a) Consider a spring of stiffness constant $\mathbf{3Nm^{-1}}$ ***connected to a body of mass*** $\mathbf{0.3\ kg}$***. It can oscillate with amplitude*** $\mathbf{0.1\ m}$***. Suppose this body executes damped SHM. Can you find the falls in energy treating it as a quantum oscillator? Explain.***

(b) Consider an atomic oscillator that can lose light energy of frequency $\mathbf{10^{14}\ s^{-1}}$***. Can you find the falls in energy treating it as a quantum oscillator? Explain.***

(c) Comment on the Planck's energy quantitation hypothesis in this regard.

⇨ (a) Consider a spring of stiffness constant $k = 3\mathrm{Nm}^{-1}$ connected to a body of. mass $m = 0.3$ kg.

Frequency of oscillation is given by $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{3\mathrm{Nm}^{-1}}{0.3\ \mathrm{kg}}} = 0.5\ \mathrm{s}^{-1}$

Total energy of oscillator, for amplitude $a = 0.1$ m is $E = \frac{1}{2}ka^2 = \frac{1}{2}\left(3\frac{N}{m}\right)(0.1\ \mathrm{m})^2 = 0.015$ J

Due to damped SHM, oscillator will lose energy.

Let us treat the oscillator as a quantum oscillator. So it will lose energy in discrete steps of amount

$\Delta E = h\nu = (6.626 \times 10^{-34})(0.5) = 3.3 \times 10^{-34}$ J then we can consider that $\Delta E = h\nu \rightarrow 0$

Clearly $\frac{\Delta E}{E} \rightarrow 0$. Energy measurements of this precision is not possible. The discrete steps through which oscillator loses energy is so small that we cannot detect it. We will find that system is losing energy not in discrete steps (or quanta) but in a continuous manner. Clearly concept of energy quantitation cannot be meaningfully applied in this problem. So instead of treating it as a quantum oscillator it is more meaningful to treat it as a classical oscillator.

Also from Planck's hypothesis related quantum number $n = \frac{E}{h\nu} = \frac{0.015}{(6.626 \times 10^{-34})(0.5)} = 4.5 \times 10^{31} \rightarrow \infty$

Hence we should use classical physics in dealing with this problem

(b) Let us treat the atomic oscillator as a quantum oscillator. So it will lose energy in discrete steps of amount $\Delta E = h\nu == (6.626 \times 10^{-34})(10^{14}) = 6.626 \times 10^{-20}$ J $= 0.4$eV

This energy is significant on an atomic scale and cannot be overlooked. So this atomic system has to be treated using quantum physics, concept of energy quantitation is meaningful here. If we apply classical Physics to describe this system, we will get erroneous results.

(c) Comment: The classical measure of mean translational energy of a particle at temperature T is kT. The quantum concept of energy quantisation involves graininess, of energy which is measured through the quantity $h\nu$ (where h is of small value = 6.626×10^{-34} J.s)

(i) For a large-scale oscillator, frequency of oscillation ν = small[$\sim 0.5\ \mathrm{s}^{-1}$ say] $\Rightarrow h\nu$ = small $\Rightarrow$ grains are of extremely small size so that the energy structure appears to be continuous (granules are overshadowed (i.e. smoothened out), they do not show up). In this case $h\nu \ll kT; \nu \to 0, T \to \infty; h$ can be ignored (i.e. h can be set to zero). This is where classical laws of physics are applicable.

(ii) For a small scale oscillator (e.g. an atomic oscillator) frequency of oscillation $\nu =$ large [$\sim 10^{14}\ \mathrm{s}^{-1}$ say] $\Rightarrow h\nu =$ large $\Rightarrow$ grains are of large size so that the energy structure appears to be discontinuous (granular structure shows up). In this case $h\nu \gg kT; \nu \to \infty, T \to 0$; we cannot ignore h. This is where classical laws of physics become inapplicable. To describe such system properly we need quantum laws of physics.

P5: A pendulum consisting of a $\mathbf{0.01\ kg}$ mass is suspended from a string $\mathbf{0.1\ m}$ in length. Let the amplitude of its oscillation be such that the string in its extreme positions makes an angle of $\mathbf{0.1}$ rad with the vertical. The energy of the pendulum decreases due, for instance, to frictional effects. Is the energy decrease observed to be continuous or discontinuous?

⇨ The oscillation frequency of the pendulum is $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{1}{2\pi}\sqrt{\frac{9.8\ \mathrm{m/sec^2}}{0.1\ \mathrm{m}}} = 1.6/\mathrm{sec}$

The energy of the pendulum is its maximum potential energy $mgh\ = mgl(1-\cos\theta)$

$\therefore mgh\ = 0.01\ \mathrm{kg} \times 9.8\ \times 0.1\ \times (1-\cos 0.1) = 5\times 10^{-5}$ joule

The energy of the pendulum is quantized so that changes in energy take place in discontinuous jumps of magnitude $\Delta E = h\nu$, but $\Delta E = h\nu = 6.63\times 10^{-34}$ joule-sec $\times 1.6/\mathrm{sec} = 10^{-33}$ joule whereas $E = 5\times 10^{-5}$ joule. Therefore, $\Delta E/E = 2\times 10^{-29}$. Hence, to measure the discreteness in the energy decrease we need to measure the energy to better than two parts in 10^{29}. It is apparent that even the most sensitive experimental equipment is totally incapable of this energy resolution.

P6: Consider a tuning fork of frequency $660\ Hz$. Treat it as a harmonic oscillator whose vibrational energy is $0.04\ J$. Compare the energy quanta of this tuning fork with those of an atomic oscillator that emits and absorbs orange light whose frequency is $5 \times 10^4\ Hz$. Discuss.

⇨ Energy quanta with frequency $v = 660$ Hz is $hv = (6.626 \times 10^{-34})(660) = 4.37 \times 10^{-31}$ J

Now energy of tuning fork treated as harmonic oscillator is $\mathrm{E} = 0.04$ J,

Then $\frac{\mathrm{E}}{\mathrm{h\nu}} = \frac{0.04\ \mathrm{J}}{4.37\times10^{-31}\ \mathrm{J}} = 9 \times 10^{28}$ hence $\Rightarrow \mathrm{E} = 9 \times 10^{28}\mathrm{h\nu}$

So energy of tuning fork is about 10^{28} times the energy quantum $\mathrm{h\nu}$. So the quantum of energy in the tuning fork is very very small and should be taken to be zero. (In fact taking $\mathrm{h\nu} \neq 0$ in this case is not a meaningful idea, as suggested from comparison with E). Hence we are justified in regarding the vibration of tuning fork to be consistent with classical physics. Here h is to be taken effectively zero.

- For the atomic oscillator, emitting and absorbing orange light of frequency, the energy quantum is $hv\ = (6.626 \times 10^{-34}\ \mathrm{J}_{-8})(5 \times 10^{14}\ \mathrm{s}^{-1}) = 3.313 \times 10^{-19}\ \mathrm{J} = 2.07\mathrm{eV}$

Now energy in the atomic scale is of the order of eV. So the atomic oscillator vibrates with energy $\sim$ eV. Clearly energy quanta and the energy of oscillation of the atomic oscillator are both $\sim$ eV and hence comparable. So treating energy of atomic oscillator as consisting of bundles of energy hv is a meaningful idea and the classical method of treating energy to be continuous is not meaningful in this situation. Here h can not be overlooked and should be considered.

P7: From Planck's radiation law A) find the total number of photons per unit volume. Given:

$\int_0^\infty \frac{x^2}{e^x-1}dx = 2.404$ *B) Find the average energy per photon.*

(c) Find the number of photons in an enclosure of volume $1cc$ *at temperature* $1000\ K$*. What is their average energy?*

⇨ A) We have Planck's radiation law for energy density of radiation between v and $v+dv$ as

$E_v dv = \frac{8\pi h v^3}{c^3}\frac{1}{e^{\frac{hv}{kT}}-1}dv$ Here hv is the energy per photon. Hence the total number of photons per unit volume in the frequency interval v and $v+dv$ given by $n_{ph}dv = \frac{E_v dv}{hv} = \frac{\frac{8\pi h v^3}{c^3}\frac{1}{e^{\frac{hv}{kT}}-1}}{hv} = \frac{8\pi v^2}{c^3}\frac{1}{hv}dv$

So for the whole frequency range, $n_{ph} = \int_0^\infty n_{ph}dv = \int_0^\infty \frac{8\pi v^2}{c^3}\frac{1}{e^{\frac{hv}{kT}}-1}dv$

Now we take $x = \frac{hv}{kT}$ so that $dv = \frac{kTdx}{h}$ Hence, $n_{ph} = \int_0^\infty \frac{8\pi\left(\frac{kTx}{h}\right)^2}{c^3}\frac{1}{e^x-1}\frac{kTdx}{h}$

$$= \int_0^\infty \frac{8\pi(kT)^3}{(hc)^3}\frac{x^2}{e^x-1}dx = \frac{8\pi(kT)^3}{(hc)^3}(2.404)$$

Putting the values of constants $k = 1.38\times10^{-23}$, $h = 6.62\times10^{-34}$ and $c = 3\times10^8$ m/s ,

we get the simplified expression of n_{ph} as $n_{ph} = 20.2\times10^6 T^3\quad m^{-3}$

B) Average energy per photon is by definition $\langle\varepsilon\rangle = \frac{\text{Total energy density of radiation (photon)(E)}}{\text{Number density of photon(n)}} = \frac{E}{n_{ph}}$

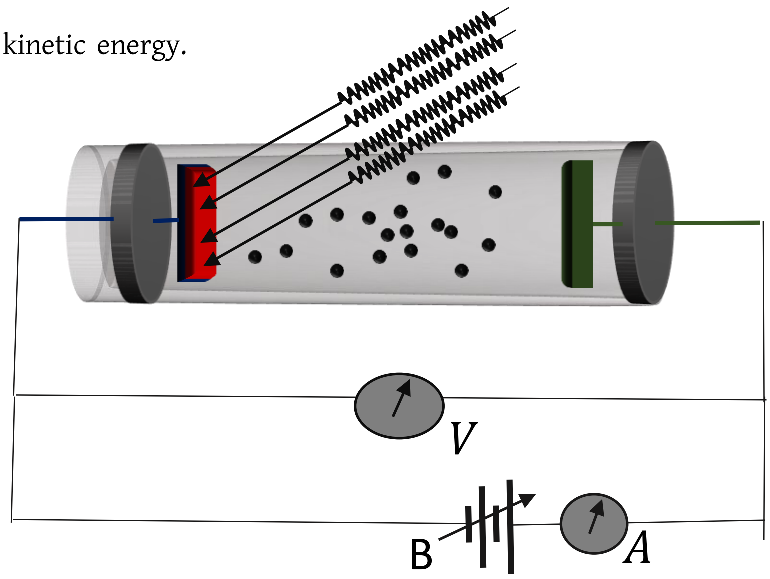
Use $E = \frac{4}{c}\sigma T^4$ where Boltzmann constant σ = 5.67×10^{-8} watt $m^{-2}K^{-4}$ and Velocity of light in free space $c = 3\times10^8$ m/s And as We know $n_{ph} = 20.2\times10^6 T^3\quad m^{-3}$

$\therefore \langle\varepsilon\rangle = \frac{E}{n_{ph}} = \frac{4}{3\times10^8}\cdot\frac{5.67\times10^{-8}T^4}{20.2\times10^6T^3} = 0.3742\times10^{-22}T$ Joule $=0.2339\times10^3$ eV.

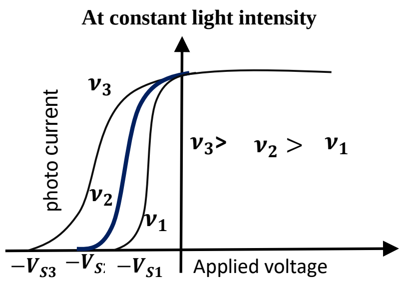
C) We know $n_{ph} = 20.2\times10^6 T^3\quad m^{-3}$ Then number of photons in an enclosure of volume 1cc at temperature 1000 K is $n = n_{ph} = 20.2\times10^6\times(1000)^3\times10^{-6} = 2.02\times10^{10}$

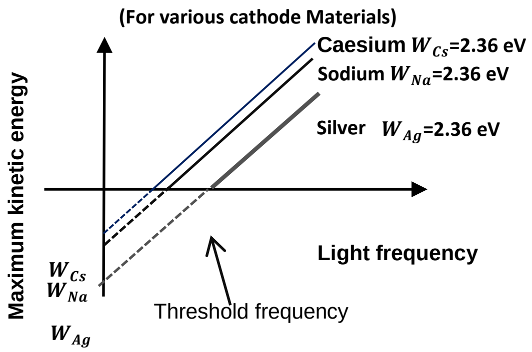
<u>THE PHOTOELECTRIC EFFECT</u>

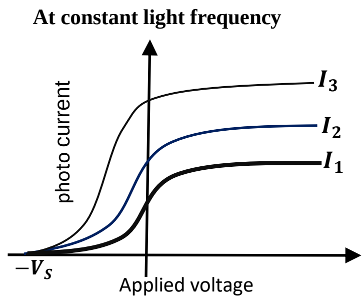
<u>The Photoelectric Effect Experiment</u> Another observation of the late 1800s that could not be understood with the classical picture of light was that certain metals, when irradiated with light, ejected electrons. The kinetic energy of the electrons depended only upon the metal and the frequency of light. Increasing the intensity of the light increases the number of electrons that are ejected, but not their kinetic energy.



For these metals, there is a minimum frequency below which no electrons are ejected, no matter how intense the light. This minimum frequency is called the threshold frequency of the metal, ν_0. As shown in Fig, a plot of the ejected electron's kinetic energy versus the frequency of the light striking the metal is a straight line.







The Photoelectric Effect Explanation

In order to explain the photoelectric effect, Albert Einstein used Planck's hypothesis that light was composed of energy quanta. He proposed that light consisted of particles of energy, which we now call *photons*, each with an energy $h\nu$. He reasoned that, if the electrons are bound to a metal by an energy, W, then the threshold frequency for electron ejection from a metal (ν_0) is that frequency for which $h\nu_0 = W$. When the frequency (energy) of the photon is less than the threshold frequency of the metal, electrons cannot be removed because the energy that binds them to the metal is greater than the energy of the photon. However, when the energy of each photon exceeds the energy that binds the electron to the metal ($h\nu > W$),

The energy of the photon is transferred to the electron. A portion of the photon's energy, *W*, is used to overcome the potential energy binding the electron to the metal and the excess energy is converted into the kinetic energy of the ejected electron (E_K).

Photo electric equation $h\nu = W + E_{K,max} = h\nu_0 + \frac{1}{2}mv_{max}^2 = h\nu_0 + eV_S$

According to Einstein, the intensity of a light wave reflects the number of photons that it contains. A beam that contains *n* photons has a total energy of $\boldsymbol{nh\nu}$, but when light interacts with matter, it does so one photon at a time. Thus, it is the energy of each photon that dictates the energy of a molecular process that is initiated by the light, while the number of events that occur per unit time, i.e., the rate of the process is dictated by the intensity of the light. Planck and Einstein had shown that light consisted of particles of energy called photons, but classical experiments showed that light is also a wave. This property of behaving as both a particle and a wave is called ***wave-particle duality.***

Expression of Planck's constant h and work function W

i) in terms of stopping potential V_0 and wavelength λ

ii) W in terms of maximum kinetic energy of photoelectrons $T_{max} \equiv T_m$ and wavelength λ

iii) in terms of maximum velocity of photoelectrons $v_{max} \equiv v_m$ and wavelength λ

Einstein's photoelectric equation is $|e|V_0 = h\nu - W$ [symbols have their usual meaning.]

Use $\nu = \frac{c}{\lambda}, c = 3 \times 10^8 \text{ ms}^{-1}$. We have then $|e|V_0 = \frac{hc}{\lambda} - W$

Let the stopping potential $V_0 = V_{01}$, for $\lambda = \lambda_1$ and $V_0 = V_{02}$, for $\lambda = \lambda_2$

Then $\therefore |e|V_{01} = \frac{hc}{\lambda_1} - W$ ....(i) and $|e|V_{02} = \frac{hc}{\lambda_2} - W$ ....(ii)

Then from equation (i) and (ii) $|e|(V_{01} - V_{02}) = hc\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = \frac{hc}{\lambda_1\lambda_2}(\lambda_2 - \lambda_1)$

Then $\boxed{h = \frac{|e|(V_{01} - V_{02})\lambda_1\lambda_2}{c(\lambda_2 - \lambda_1)}}$ ....(iii)

Again multiplying equation (i) by $\frac{1}{\lambda_2}$ and (ii) by $\frac{1}{\lambda_1}$ we get $|e|\frac{V_{01}}{\lambda_2} = \frac{hc}{\lambda_1\lambda_2} - \frac{W}{\lambda_2}$ ....(iv)

And $|e|\frac{V_{02}}{\lambda_1} = \frac{hc}{\lambda_1\lambda_2} - \frac{W}{\lambda_1}$ ....(v)

Hence we get from equation (iii) and (iv) $|e|\left(\frac{V_{01}}{\lambda_2} - \frac{V_{02}}{\lambda_1}\right) = W\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$ [subtracting]

Hence $|e|(V_{01}\lambda_1 - V_{02}\lambda_2) = W(\lambda_2 - \lambda_1)$ or, $\boxed{W = \frac{|e|(V_{01}\lambda_1 - V_{02}\lambda_2)}{\lambda_2 - \lambda_1}}$..............(vi)

(ii) Let maximum kinetic energy of photoelectrons $T_m = T_{m1}$, for $\lambda = \lambda_1$ and $T_m = T_{m1}$, for $\lambda = \lambda_1$

Using $|e|\, V_{01} = T_{m1}$ & $|e|V_{02} = T_{m2}$ we get $\boxed{h = \frac{(T_{m1} - T_{m2})\lambda_1\lambda_2}{c(\lambda_2 - \lambda_1)}}$ and $\boxed{W = \frac{T_{m1}\lambda_1 - T_{m2}\lambda_2}{\lambda_2 - \lambda_1}}$

Let maximum velocity of photoelectrons $T_m = T_{m1}$, for $\lambda = \lambda_1$ and $T_m = T_{m1}$, for $\lambda = \lambda_1$

Using $T_{m1} = \frac{1}{2}mv_{m1}^2$ & $T_{m2} = \frac{1}{2}mv_{m2}^2$ we get $\boxed{h = \frac{m(v_{m1}^2 - v_{m2}^2)\lambda_1\lambda_2}{2c(\lambda_2 - \lambda_1)}}$ and $\boxed{W = \frac{m(v_{m1}^2\lambda_1 - v_{m2}^2\lambda_2)}{2(\lambda_2 - \lambda_1)}}$

P8: The threshold wavelength of photoelectric material is $\mathbf{2300\AA}$*. An ultraviolet light of wavelength* $\mathbf{1800\AA}$ *is incident on it. What will be the kinetic energy of ejected electron?*

⇨ The energy of incident photon is $\frac{hc}{\lambda}=\frac{6.62\times10^{-34}\times3\times10^{8}}{1800\times10^{-10}}=1.1\times10^{-18}$ J

The work function $W=\frac{hc}{\lambda_0}=\frac{6.62\times10^{-34}\times3\times10^{8}}{2300\times10^{-1}}=0.86\times10^{-18}$ J

So the kinetic energy of ejected electron $\frac{1}{2}mv_{\max}^2=$ incident energy − work function

Then $\frac{1}{2}mv_{\max}^2=1.1\times10^{-18}J-0.86\times10^{-18}J=0.24\times10^{-18}J$

P9: The stopping potential is $\mathbf{2.72}$ *Volts for wavelength 2000 Å for copper material. What is the work function of copper?*

⇨ We have from Einstein's equation, $eV_S=hv-w_0$ or, $w_0=\frac{hc}{\lambda}-eV_s$

Then $w_0=\frac{6.62\times10^{-34}\times3\times10^{8}}{2000\times10^{-1}}-1.6\times10^{-19}\times2.72=5.58\times10^{-19}J=3.49\text{eV}.$

P10: Light of wavelength $\mathbf{2000\AA}$ *falls on a metallic surface. If the work function of the surface* $\mathbf{4.2eV}$*, what is the kinetic energy of the fastest photoelectrons emitted? Also calculate the stopping potential and the threshold wavelength for the metal.*

⇨ The energy of the radiation having wavelength 2000Å is obtained as

$$\frac{hc}{\lambda}=\frac{(6.626\times10^{-34}\text{ J s})(3\times10^{8}\text{ m/s})}{(2000\times10^{-10}\text{ m})(1.6\times10^{-19}\text{ J/eV})}=6.212\text{eV}$$

Work function $=4.2$ eV Then KE of fastest electron $=6.212-4.2=2.012$eV

Stopping potential $=2.012$ V

Threshold wavelength $\lambda_0=\frac{hc}{\text{Work function}}$ Then $\lambda_0=\frac{(6.626\times10^{-34}\text{ J s})(3\times10^{8}\text{ m/s})}{(4.2\text{eV})(1.6\times10^{-19}\text{ J/eV})}=2958\text{Å}$

P11: When radiation of wavelength $1500\AA$ is incident on a photocell, . If the stopping potential is 4.4 volts, calculate the work function, threshold frequency and threshold wavelength.

⇨ Energy of the incident photon $= \frac{hc}{\lambda} = \frac{(6.626\times10^{-34}\text{ J s})(3\times10^{8}\text{ m/s})}{(1500\times10^{-10}\text{ m})(1.6\times10^{-19}\text{ J/eV})} = 8.28\text{eV}$

Work function $= 8.28 - 4.4 = 3.88\text{eV}$ Threshold frequency $\nu_0 = \frac{3.88\text{eV}(1.6\times10^{-19}\text{ J/eV})}{6.626\times10^{-34}\text{ J s}} = 9.4\times10^{14}\text{ Hz}$

Threshold wavelength $\lambda_0 = \frac{c}{\nu_0} = \frac{3\times10^{8}\text{ m/s}}{9.4\times10^{14}\text{ s}^{-1}} = 3191\text{Å}$

P12: The work function of barium and tungsten are $2.5eV$ and $4.2eV$, respectively. Check whether these materials are useful in a photocell, which is to be used to detect visible light.

⇨ The wavelength λ of visible light is in the range $4000 - 7000\text{Å}$. Then,

Energy of 4000Å light $= \frac{hc}{\lambda} = \frac{(6.626\times10^{-34}\text{ J s})(3\times10^{8}\text{ m/s})}{(4000\times10^{-10}\text{ m})(1.6\times10^{-19}\text{ J/eV})} = 3.106\text{eV}$

Energy of 7000Å light $= \frac{6.626\times10^{-34}\times3\times10^{8}}{7000\times10^{-10}\times1.6\times10^{-19}} = 1.77\text{eV}$

The work function of tungsten is 4.2eV, which is more than the energy range of visible light. Hence. barium is the only material useful for the purpose.

P13: When light of wavelength $4800\AA$ is used to irradiate a photocathode the emitted photo electrons are found to have a stopping potential of 0.59 volts. With light of $3200\AA$, the stopping potential is 1.87 volts. Calculate (i) Planck's constant $\mathbf{h}$ and (ii) Work function $\mathbf{W}$.

As $h = \frac{|e|(V_{01}-V_{02})\lambda_1\lambda_2}{c(\lambda_2-\lambda_1)} = \frac{1.6\times10^{-19}(1.87-0.59)4800\times10^{-10}\times3200\times10^{-10}}{3\times10^{8}(4800-3200)\times10^{-10}} = 65.63\times10^{-35} = 6.563\times10^{-34}\text{ J s}$

And $W = \frac{|e|(V_{01}\lambda_1-V_{02}\lambda_2)}{\lambda_2-\lambda_1} = \frac{1.6\times10^{-19}(1.87\times4800\times10^{-10}-0.59\times3200\times10^{-10})}{(4800-3200)\times10^{-10}} = 1.6\times10^{-19}\times4.43\,J = 4.43\text{ eV}.$

A. COMPTON EFFECT

In 1921 Prof A.H. Compton discovered that when a monochromatic beam of high frequency (or, low wavelength) radiation (e.g. X-rays, γ-rays) is scattered by a substance, the scattered radiation contains the radiation of lower frequency (or, greater wavelength) along with the radiation of unchanged frequency (or, unchanged wavelength)

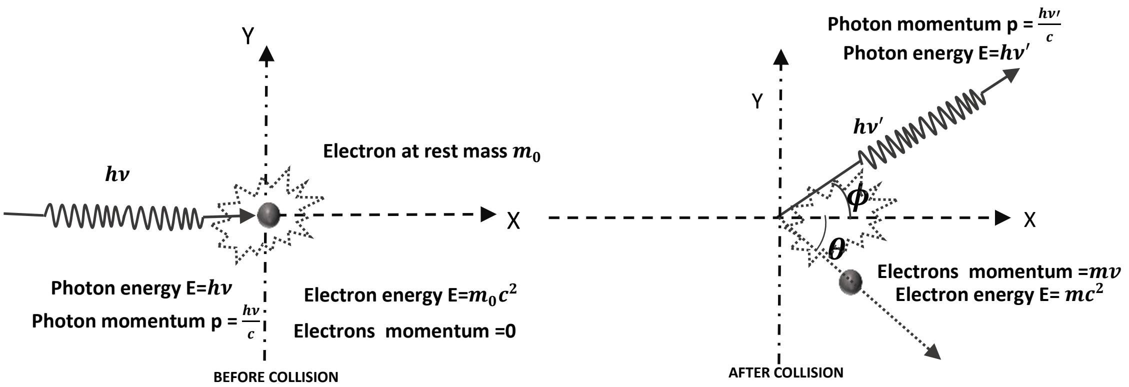
This phenomenon is known as **Compton Effect.**

This effect was explained on the basis of quantum theory of light, according to which the radiations consists of photon each having energy $h\nu$. When a photon of energy $h\nu$ collides with free electron of the scattering substance initially at rest. It transfers some energy to electron. Due to this energy the electron gain K.E and recoil with velocity v. Hence scattered photon will have lower energy. i.e, lower frequency or, greater wavelength than incident one.

Assumptions – 1. **Compton Effect is a result of interaction of a individual photon and free electron as target.**

2. The collision is relativistic and elastic.

3. The laws of conservation of linear momentum and energy holds.



We have from **energy conservation** law $\boldsymbol{h\nu + m_0c^2 = h\nu' + mc^2}$**……………………………………(I)**

We have from **momentum conservation law** $\boldsymbol{\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\phi + mv\cos\theta}$ **……………..along X-axis**

$\boldsymbol{0 = \frac{h\nu'}{c}\sin\phi - mv\sin\theta}$ **……………..along Y-axis** **…………..(II)**

From equation (II) we have $mvc\cos\theta = h\nu - h\nu'\cos\phi$

& $mvc\sin\theta = h\nu'\sin\phi$**…………………………(III)**

From equation (III) we have $\boldsymbol{m^2v^2c^2 = h^2(\nu^2 + \nu'^2 - 2\nu\nu'\cos\phi)}$**……………………………(IV)**

From equation (I) we have $\boldsymbol{mc^2 = h(\nu - \nu') + m_0c^2}$

or, $m^2c^4 = h^2(\boldsymbol{\nu - \nu'})^2 + {m_0}^2c^4 + 2\,\boldsymbol{h(\nu - \nu')m_0c^2}$

or, $m^2c^4 = h^2(\nu^2 + \nu'^2 - 2\nu\nu') + {m_0}^2c^4 + 2\,\boldsymbol{h(\nu - \nu')m_0c^2}$**………………………………………………(v)**

From eqn. **(IV) & (V)** We $\cancel{m^2c^4\left(1-\frac{v^2}{c^2}\right)} = -2\nu\nu'\boldsymbol{h^2}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi}) + \cancel{m_0{}^2c^4} + 2\,\boldsymbol{h}(\boldsymbol{\nu}-\boldsymbol{\nu}')\boldsymbol{m_0c^2}$

$$\left[\textbf{As } \boldsymbol{m} = \frac{\boldsymbol{m_0}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}\right]$$

Or, $\frac{m_0c^2}{h}\frac{(\nu-\nu')}{\nu\nu'} = (\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi})$ **or,** $\frac{c(\nu-\nu')}{\nu\nu'} = \frac{h}{m_0c}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi})$ **or,** $\frac{c}{\nu'}-\frac{c}{\nu} = \frac{h}{m_0c}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi})$

Or, $\boldsymbol{\lambda'}-\boldsymbol{\lambda} = \frac{h}{m_0c}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi})$

or, Compton shift= $\Delta\boldsymbol{\lambda} = \boldsymbol{\lambda_C}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi})$

[where $\frac{h}{m_0c} = \lambda_C = Compton$ wavelength of electron=0.024 Å]

Note that -1**. Change in wavelength is independent of wavelength of incident radiation and scattering material. It depends only on scattering angle.**

2. Maximum value of $(1-\cos\phi) = 2$, hence maximum change in wavelength is 0.024× $2 = 0.048$ Å

3. For vissible ligth (say $\lambda \approx 5000$ Å) the maximum change in wavelength ≈ 0.05Å is only about 0.001% Which is undetectable. Hence Compton Effect cannot be observed for vissible light.

4. If the electron is tightly bound to the atom, atoms as a whole recoils as a result of Compton sacttering. Say for aluminium terget Compton wavelength $\lambda_c = \frac{h}{m_{Al}c} = \frac{h}{27\times1840\times m_0c} = \frac{0.024}{27\times1840} = 4.9\times10^{-7}$ Å. Hence change of wavelegth is negligible for all values of ϕ. This explain the presence of unmodified radiation.

❖ **Relation between angle of scattering and angle at which recoil electron emitted.**

We have the relation $\frac{c}{\nu'}-\frac{c}{\nu} = \frac{h}{m_0c}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi})$ **or,** $\frac{1}{\nu'} = \frac{1}{\nu}+\frac{h}{m_0c^2}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi}) = \frac{1}{\nu}\left(\mathbf{1}+\frac{h\nu}{m_0c^2}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi})\right)$

or, $\frac{\nu}{\nu'} = (\mathbf{1}+\boldsymbol{\alpha}(\mathbf{1}-\mathbf{cos}\,\boldsymbol{\phi}))$ **[where** $\frac{h\nu}{m_0c^2} = \boldsymbol{\alpha}$**]**

$\therefore\ \frac{\nu}{\nu'} = \left[\mathbf{1}+\mathbf{2}\boldsymbol{\alpha}\,\mathbf{sin}^2\frac{\phi}{2}\right]$ **……………………………………(i)**

now $m\nu c\cos\theta = h\nu - h\nu'\cos\phi$ & $m\nu c\sin\theta = h\nu'\sin\phi$ then, $\cot\theta = \frac{h\nu-h\nu'\cos\phi}{h\nu'\sin\phi} = \frac{\nu-\nu'\cos\phi}{\nu'\sin\phi} = \frac{\frac{\nu}{\nu'}-\cos\phi}{\sin\phi}$

$\therefore\ \cot\theta = \frac{1+2\alpha\sin^2\frac{\phi}{2}-\cos\phi}{\sin\phi} = \frac{2\alpha\sin^2\frac{\phi}{2}+1-\cos\phi}{\sin\phi} = \frac{2\alpha\sin^2\frac{\phi}{2}+2\sin^2\frac{\phi}{2}}{\sin\phi} = (1+\alpha)\frac{2\sin^2\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = (1+\alpha)\tan\frac{\phi}{2}$

$$\therefore\ \mathbf{cot}\,\boldsymbol{\theta} = (\mathbf{1}+\boldsymbol{\alpha})\,\mathbf{tan}\frac{\boldsymbol{\phi}}{\mathbf{2}}$$

❖ **The kinetic energy of recoil electron.**

The kinetic energy of recoil electron is given by ,

$T_e = h\nu - h\nu' = h\nu\left(1-\frac{\nu'}{\nu}\right) = h\nu\left(1-\frac{1}{1+2\alpha\sin^2\frac{\phi}{2}}\right) = h\nu\left(\frac{2\alpha\sin^2\frac{\phi}{2}}{1+2\alpha\sin^2\frac{\phi}{2}}\right)$ **[where** $\frac{h\nu}{m_0c^2} = \boldsymbol{\alpha}$ **]**

Or, $T_e = h\nu\left(1-\frac{1}{(1+\alpha(1-\cos\phi))}\right) = h\nu\left(\frac{\alpha(1-\cos\phi)}{(1+\alpha(1-\cos\phi))}\right)$

Note that maximun kinetic energy = $(T_e)_{max} = h\nu\left(\frac{2\alpha}{1+2\alpha}\right) = \frac{h\nu}{1+\frac{1}{2\alpha}}$ i.e. when **ϕ**=180°

As maximum K.E of the recoil photon is $\frac{h\nu}{1+\frac{1}{2\alpha}}$ then $(T_e)_{max}$ is always $< h\nu$.

Hence electron can get a fraction of energy of incident photon or, photon can never transfer its hole energy to the electron.

❖ **Show that a free electron at rest cannot absorb a photon in Compton Effect**

If P_E & P_P are the momentum and E_E & E_P are energies of free electon and incident photon. If absorbtion is possible then $P_E = P_P$ & $E_E = E_P$.

Now if $E_E = E_P$ then, $\sqrt{(cP_E)^2 + (m_0c^2)^2} = h\nu$ or, $\sqrt{(P_E)^2 + (m_0c)^2} = h\nu/c$ or, $P_E < P_P$

Hence conservation of momentum breaks down. Hence a free electron cannot absorb a photon. This justifies that for Compton Effect to be occur the electron must be free.

❖ **Explain why an electron in Compton Effect cannot be scattered at an angle greater than 90°**

We have the relation $\cot\theta = (1+\alpha)\tan\frac{\phi}{2}$ when ϕ=0 then $\boldsymbol{\theta}=\frac{\pi}{2}$ and When $\phi=\pi$ then $\boldsymbol{\theta}$=0

Then for the whole range $0 \le \phi \le \pi$ we have $0 \le \theta \le \frac{\pi}{2}$ then an electron in Compton Effect cannot be scattered at an angle greater than 90°.