Problems and solutions

(Wave Mechanical Concepts)

Wave Nature of Particles

Classical physics considered particles and waves as distinct entities. Quantum ideas firmly established that radiation has both wave and particle nature. This dual nature was extended to material particles by Louis de Broglie in 1924. The wave associated with a particle in motion, called matter wave, has the wavelength λ given by the de Broglie equation

$$\lambda = \frac{h}{p} = \frac{h}{mv} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where p is the momentum of the particle. Electron diffraction experiments conclusively proved the dual nature of material particles in motion.

Uncertainty Principle

When waves are associated with particles, some kind of indeterminacy is bound to be present. Heisenberg critically analyzed this and proposed the uncertainty principle:

$$\Delta x \cdot \Delta p_x \simeq h \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

where Δx is the uncertainty in the measurement of position and Δp_x is the uncertainty in the measurement of the x-component of momentum. A more rigorous derivation leads to

$$\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2} \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Two other equally useful forms are the energy time and angular momentum-polar angle relations given respecting by

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

$$\Delta L_z \cdot \Delta\phi \geq \frac{\hbar}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

Wave Packet

The linear superposition principle, which is valid for wave motion, is also valid for material particles. To describe matter waves associated with particles in motion, we require a quantity which varies in space and time. This quantity, called the wave function $\Psi(r, t)$, is confined to a small region in space and is called the wave packet or wave group. M athematically, a wave packet can be constructed by the superposition of an infinite number of plane waves with slightly differing k-values, as

$$\Psi(x,t) = \int A(k)\exp[ikx - i\omega(k)t]dk \ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

where k is the wave vector and ω is the angular frequency. Since the wave packet is localized, the limit of the integral is restricted to a small range of k-values, say, $(k_0 - \Delta k) < k < (k_0 + \Delta k)$. The speed with which the component waves of the wave packet move is called the phase velocity $\mathbf{v}_\mathrm{p}$ which is defined as

$$\mathbf{v}_\mathrm{p} = \frac{\omega}{\mathrm{k}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vii)}$$

The speed with which the envelope of the wave packet moves is called the group velocity $\mathbf{v}_g$ given by $\mathbf{v}_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{dk}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(viii)}$

Time-dependent Schrödinger Equation

For a detailed study of systems, Schrödinger formulated an equation of motion for $\Psi(\boldsymbol{r}, \mathrm{t})$:

$$\mathrm{i}\hbar \frac{\partial}{\partial \mathrm{t}} \Psi(\boldsymbol{r}, \mathrm{t}) = \left[-\frac{\hbar}{2\ \mathrm{m}} \nabla^2 + \mathrm{V}(\mathrm{r})\right] \Psi(\boldsymbol{r}, \mathrm{t}) \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(ix)}$$

The quantity in the square brackets is called the Hamiltonian operator of the system. Schrödinger realized that, in the new mechanics, the energy E, the momentum $\boldsymbol{p}$, the coordinate r, and time t have to be considered as operators operating on functions. A n analysis leads to the following operators for the different dynamical variables:

$$\mathrm{E} \rightarrow \mathrm{i}\hbar \frac{\partial}{\partial \mathrm{t}},\ \mathrm{p} \rightarrow -\mathrm{i}\hbar\nabla,\ \boldsymbol{r} \rightarrow \boldsymbol{r},\ \mathrm{t} \rightarrow \mathrm{t} \ldots\ldots\ldots\ldots\ldots\ldots \text{(x)}$$

Physical Interpretation of $\Psi(r, t)$

Probability Interpretation

A universally accepted interpretation of $\Psi(\boldsymbol{r}, \mathrm{t})$ was suggested by Born in 1926. He interpreted $\Psi^*\Psi$ as the position probability density $P(\boldsymbol{r}, \mathrm{t})$:

$$P(\boldsymbol{r}, \mathrm{t}) = \Psi^*(\boldsymbol{r}, \mathrm{t})\Psi(\boldsymbol{r}, \mathrm{t}) = |\Psi(\boldsymbol{r}, \mathrm{t})|^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(xi)}$$

The quantity $|\Psi(\boldsymbol{r}, \mathrm{t})|^2\ \mathrm{d}\tau$ is the probability of finding the system at time t in the elementary volume $\mathrm{d}\tau$ surrounding the point $\boldsymbol{r}$. Since the total probability is 1, we have

$$\int_{\infty}^{\infty} |\Psi(\boldsymbol{r}, \mathrm{t})|^2\ \mathrm{d}\tau = 1 \ldots\ldots\ldots\ldots\ldots\ldots \text{(xii)}$$

If Ψ is not satisfying this condition, one can multiply Ψ by a constant, say N, so that $N\Psi$ satisfies Eq. (xii). Then,

$$|\mathrm{N}|^2 \int_{\infty}^{\infty} |\Psi(\boldsymbol{r}, \mathrm{t})|^2\ \mathrm{d}\tau = 1 \ldots\ldots\ldots\ldots\ldots\ldots \text{(xiii)}$$

The constant N is called the normalization constant.

Probability Current Density

The probability current density $\boldsymbol{j}(\boldsymbol{r}, \mathrm{t})$ is defined as

$$\boldsymbol{j}(\boldsymbol{r}, \mathrm{t}) = \frac{\mathrm{i}\hbar}{2\ \mathrm{m}}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)\ldots\ldots\ldots\ldots\ldots\ldots \text{(xiv)}$$

It may be noted that, if Ψ is real, the vector $\boldsymbol{j}(\boldsymbol{r}, \mathrm{t})$ vanishes. The function $\boldsymbol{j}(\boldsymbol{r}, \mathrm{t})$ satisfies the equation of continuity

$$\frac{\partial}{\partial \mathrm{t}}\mathrm{P}(\boldsymbol{r}, \mathrm{t}) + \nabla \cdot \boldsymbol{j}(\boldsymbol{r}, \mathrm{t}) = 0\ldots\ldots\ldots\ldots\ldots\ldots \text{(xv)}$$

Equation (xv) is a quantum mechanical probability conservation equation. That is, if the probability of finding the system in some region increases with time, the probability of finding the system outside decreases by the same amount.

Time-independent Schrödinger Equation

If the Hamiltonian operator does not depend on time, the variables r and t of the wave function $\Psi(\boldsymbol{r}, \mathrm{t})$ can be separated into two functions $\psi(\boldsymbol{r})$ and $\phi(\mathrm{t})$ as

$$\Psi(\boldsymbol{r}, \mathrm{t}) = \psi(\boldsymbol{r})\phi(\mathrm{t})\ \ldots\ldots\ldots\ldots\ldots\ldots \text{(xvi)}$$

Simplifying, the time-dependent Schrödinger equation, Eq. (2.9), splits into the following two equations:

$$\frac{1}{\phi(\mathrm{t})}\frac{\mathrm{d}\phi}{\mathrm{dt}} = -\frac{\mathrm{iE}}{\hbar}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(xvii)$$

$$\left[-\frac{\hbar}{2\ \mathrm{m}}\nabla^2 + \mathrm{V(r)}\right]\psi(\mathrm{r}) = \mathrm{E}\psi(\mathrm{r})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(xviii)$$

The separation constant E is the energy of the system. Equation ($xviii$) is the time-independent Schrödinger equation. The solution of Eq. ($xvii$) gives

$$\phi(\mathrm{t}) = \mathrm{Ce}^{-\mathrm{iEt}/\hbar}\ldots\ldots\ldots\ldots\ldots\ldots \text{(xix)}$$

where C is a constant. $\Psi(\boldsymbol{r}, \mathrm{t})$ now takes the form

$$\Psi(\boldsymbol{r}, \mathrm{t}) = \psi(\boldsymbol{r})\mathrm{e}^{-\mathrm{iEt}/\hbar}\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{xx})$$

The states for which the probability density is constant in time are called stationary states, i.e.,

$$P(\boldsymbol{r}, \mathrm{t}) = |\Psi(\boldsymbol{r}, \mathrm{t})|^2 = \text{ constant in time } \ldots\ldots\ldots\ldots\ldots\ldots \text{(xxi)}$$

<u>Admissibility conditions on the wave functions</u>

(i) The wave function $\Psi(\boldsymbol{r}, \mathrm{t})$ must be finite and single valued at every point in space.

(ii) The functions Ψ and $\nabla\psi$ must be continuous, finite and single valued.

PROBLEMS

P/1 Calculate the de Broglie wavelength of an electron having a kinetic energy of 1000eV. Compare the result with the wavelength of x-rays having the same energy.

Solution. ⇨ The momentum $p = \sqrt{2\,m\,E}$

Then, $\lambda = \frac{\text{h}}{\text{p}} = \frac{6.626\times10^{-34}\text{js}}{[2\times(9.11\times10^{-31}\text{ kg})\times(1.6\times10^{-16}\text{ J}]^{1/2}} = 0.39\times10^{-10}\text{ m} = 0.39\text{Å}$

For x-rays, Energy $= h\nu \; = \frac{hc}{\lambda}$, then $\lambda = \frac{hc}{E}$

$$\lambda = \frac{(6.626\times10^{-34}\text{ J s})\times(3\times10^{8}\text{ m/s})}{1.6\times10^{-16}\text{ J}} = 12.42\times10^{-10}\text{ m} = 12.42\text{Å}$$

$$\frac{\text{Wavelength of } x\text{-rays}}{\text{de B roglie wavelength of electron}} = \frac{12.42\text{Å}}{0.39\text{Å}} = 31.85$$

P/2 Determine the de Broglie wavelength of an electron that has been accelerated through a potential difference of (i) 100 V, (ii) 200 V.

Solution. ⇨ (i) The energy gained by the electron $= 100\text{eV}$. Then,

$$\frac{p^2}{2\text{ m}} = 100\text{eV} \quad = (100\text{eV})(1.6\times10^{-19}\text{ J/eV}) = 1.6\times10^{-17}\text{ J}$$

$$\begin{aligned}
\text{p} &= [2(9.1\times10^{-13}\text{ kg})(1.6\times10^{-17}\text{ J})]^{1/2}\\
&= 5.396\times10^{-24}\text{ kg ms}^{-1}\\
\lambda &= \frac{\text{h}}{\text{p}} = \frac{6.626\times10^{-34}\text{ J}}{5.396\times10^{-24}\text{ kg ms}^{-1}}\\
&= 1.228\times10^{-10}\text{ m} = 1.128\text{Å}\\
&\frac{\text{p}^2}{2\text{ m}} = 200\text{eV} = 3.2\times10^{-17}\text{ J}\\
\text{p} &= [2(9.1\times10^{-31}\text{ kg})(3.2\times10^{-17}\text{ J})]^{1/2}\\
&= 7.632\times10^{-24}\text{ kg ms}^{-1}\\
\lambda &= \frac{\text{h}}{\text{p}} = \frac{6.626\times10^{-34}\text{ J s}}{7.632\times10^{-24}\text{ kg ms}^{-1}}\\
&= 0.868\times10^{-10}\text{ m} = 0.868\text{Å}
\end{aligned}$$

P/3 The electron scattering experiment gives a value of $2\times10^{-15}\text{ m}$ for the radius of a nucleus. Estimate the order of energies of electrons used for the experiment. Use relativistic expressions.

Solution. ⇨ For electron scattering experiment, the de Broglie wavelength of electrons used must be of the order of $4\times10^{-15}\text{ m}$, the diameter of the atom. The kinetic energy

$$T = E - m_0^2c^2 = \sqrt{c^2p^2 + m_0^2c^4} - m_0c^2$$
$$(T + m_0c^2)^2 = c^2p^2 + m_0^2c^4$$
$$m_0^2c^4\left(1 + \frac{T}{m_0c^2}\right)^2 = c^2p^2 + m_0^2c^4$$
$$c^2p^2 = m_0^2c^4\left[\left(1 + \frac{T}{m_0c^2}\right)^2 - 1\right]$$
$$p = m_0c\left[\left(1 + \frac{T}{m_0c^2}\right)^2 - 1\right]^{1/2}$$
$$\frac{h^2}{\lambda^2} = m_0^2c^2\left[\left(1 + \frac{T}{m_0c^2}\right)^2 - 1\right]$$
$$\left(1 + \frac{T}{m_0c^2}\right)^2 = \frac{h^2}{\lambda^2 m_0^2 c^2} + 1$$
$$= \frac{(6.626 \times 10^{-34}\text{Js})^2}{(16 \times 10^{-30}\ \text{m}^2) \times (9.11 \times 10^{-31}\ \text{kg})^2 \times (3 \times 10^8\ \text{m/s})^2} + 1$$
$$= 3.6737 \times 10^5$$
$$\text{T} = 605.1\ \text{m}_0\text{c}^2$$
$$= 605.1 \times (9.11 \times 10^{-31}\ \text{kg}) \times (3 \times 10^8\ \text{m/s})^2$$
$$= 496.12 \times 10^{-13}\ \text{J} = \frac{496.12 \times 10^{-13}\ \text{J}}{1.6 \times 10^{-19}\ \text{J/eV}}$$
$$= 310 \times 10^6\text{eV} = 310\text{MeV}$$

P/4 Evaluate the ratio of the de Broglie wavelength of electron to that of proton when (i) both have the same kinetic energy, and (ii) the electron kinetic energy is 1000eV and the proton KE is 100eV.

Solution. ⇨ (i) $\lambda_1 = \frac{\text{h}}{\sqrt{2m_1T_1}}$; $\lambda_2 = \frac{\text{h}}{\sqrt{2m_2T_2}}$; $\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{m_2T_2}{m_1T_1}}$

$$\frac{\lambda \text{ of electron}}{\lambda \text{ of proton}} = \sqrt{\frac{1836m_eT}{m_eT}} = \sqrt{1836} = 42.85$$

(ii) $\text{T}_1 = 1000\text{eV}; \text{T}_2 = 100\text{eV}$

$$\frac{\lambda \text{ of electron}}{\lambda \text{ of proton}} = \sqrt{\frac{1836 \times 100}{1000}} = 13.55$$

P/5 Proton beam is used to obtain information about the size and shape of atomic nuclei. If the diameter of nuclei is of the order of 10^{-15} m, what is the approximate kinetic energy to which protons are to be accelerated? Use relativistic expressions.

Solution. ⇨ When fast moving protons are used to investigate a nucleus, its de Broglie wavelength must be comparable to nuclear dimensions, i.e., the de Broglie wavelength of protons must be of the order of 10^{-15} m. In terms of the kinetic energy T, the relativistic momentum p is given by (refer Problem 2.3)

$$p = m_0 c \sqrt{\left(1 + \frac{T}{m_0 c^2}\right) - 1},\ \lambda = \frac{h}{p} \cong 10^{-15}\ \mathrm{m}$$

$$\frac{h^2}{\lambda^2} = m_0^2 c^2 \left[\left(1 + \frac{T}{m_0 c^2}\right)^2 - 1\right]$$

Substitution of λ_1 $\mathrm{m_0}$, h and c gives $\quad \mathrm{T} = 9.8912 \times 10^{-11}\ \mathrm{J} = 618.2\mathrm{MeV}$

P/6 Estimate the velocity of neutrons needed for the study of neutron diffraction of crystal structures if the interatomic spacing in the crystal is of the order of 2Å. Also estimate the kinetic energy of the neutrons corresponding to this velocity. M ass of neutron $= 1.6749 \times 10^{-27}$ kg.

Solution. ⇨ de Broglie wavelength $\lambda \cong 2 \times 10^{-10}$ m

$$\lambda = \frac{\mathrm{h}}{\mathrm{mv}} \text{ or } \mathbf{v} = \frac{\mathrm{h}}{\mathrm{m}\lambda} = \frac{6.626 \times 10^{-34}\ \mathrm{J}}{(1.6749 \times 10^{-27}\ \mathrm{kg})(2 \times 10^{-10}\ \mathrm{m/s})} = 1.978 \times 10^{3}\ \mathrm{ms^{-1}}$$

$$\text{Kinetic energy } T = \frac{1}{2}\mathrm{mv}^2 = \frac{1}{2}(1.6749 \times 10^{-27}\ \mathrm{kg})(1.978 \times 10^{3}\ \mathrm{ms^{-1}})^2$$
$$= 3.2765 \times 10^{-21}\ \mathrm{J} = 20.478 \times 10^{-3}\mathrm{eV}$$

P/7 Estimate the energy of electrons needed for the study of electron diffraction of crystal structures if the interatomic spacing in the crystal is of the order of 2Å. Solution. de Broglie wavelength of electrons ≅ 2Å $= 2 \times 10^{-10}$ m

Solution. ⇨ kinetic energy $T == \frac{p^2}{2\mathrm{\ m}} = \frac{(\mathrm{h}/\lambda)^2}{2\mathrm{\ m}} = \frac{(6.626\times10^{-34}\mathrm{Js})^2}{2\times(2\times10^{-10}\ \mathrm{m})^2(9.11\times10^{-31}\ \mathrm{kg})}$

$$= 60.24 \times 10^{-19}\ \mathrm{J} = 37.65\mathrm{eV}$$

P/8 What is the ratio of the kinetic energy of an electron to that of a proton if their de Broglie wavelengths are equal?

Solution. ⇨

$\mathrm{m_1} =$ mass of electron,
$m_2 =$ mass of proton,
$\mathbf{v}_1 =$ velocity of electron,
$\mathbf{v}_2 =$ velocity of proton .

$$\lambda = \frac{h}{m_1 \mathbf{v}_1} = \frac{h}{m_2 \mathbf{v}_2} \text{ or } m_1 \mathbf{v}_1 = m_2 \mathbf{v}_2$$

$$m_1\left(\frac{1}{2} m_1 \mathbf{v}_1^2\right) = m_2\left(\frac{1}{2} m_2 \mathbf{v}_2^2\right)$$

$$\frac{\text{Kinetic energy of electron}}{\text{Kinetic energy of proton}} = \frac{m_2}{m_1} = 1836$$

P/9 A electron of rest mass m_0 is accelerated by an extremely high potential of V volts. Show that its wavelength $\lambda = \frac{\mathrm{hc}}{[\mathrm{eV}(\mathrm{eV}+2\ \mathrm{m}_0 c^2)]^{1/2}}$

Solution. ⇨ The energy gained by the electron in the potential is Ve. The relativistic expression for kinetic energy $= \frac{m_0c^2}{(1-v^2/c^2)^{1/2}} - m_0c^2$.

Equating the two and rearranging, we get $\frac{m_0c^2}{(1-v^2/c^2)^{1/2}} - m_0c^2 = Ve$

$$\text{or,}\quad (1-\mathbf{v}^2/c^2)^{1/2} = \frac{m_0c^2}{Ve+m_0c^2}$$

$$1-\frac{\mathbf{v}^2}{c^2} = \frac{m_0^2c^4}{(Ve+m_0c^2)^2}$$

$$\frac{\mathbf{v}^2}{c^2} = \frac{(Ve+m_0c^2)^2 - m_0^2c^4}{(Ve+m_0c^2)^2} = \frac{Ve(Ve+2m_0c^2)}{(Ve+m_0c^2)^2}$$

$$\mathbf{v} = \frac{c[Ve(Ve+2m_0c^2)]^{1/2}}{Ve+m_0c^2}$$

de Broglie Wavelength

$$\lambda = \frac{h}{m\mathbf{v}} = \frac{h(1-\mathbf{v}^2/c^2)^{1/2}}{m_0\mathbf{v}}$$

$$\lambda = \frac{h}{m_0}\frac{m_0c^2}{Ve+m_0c^2}\frac{Ve+m_0c^2}{c[Ve(Ve+2m_0c^2)]^{1/2}}$$

$$= \frac{hc}{[Ve(Ve+2m_0c^2)]^{1/2}}$$

P/10 Show that the phase velocity $\mathbf{v}_p$ for a particle with rest mass m_0 is always greater than the velocity of light and that $\mathbf{v}_p$ is a function of wavelength.

Solution. ⇨ Combining the two, we get Phase velocity $\mathbf{v}_p = \frac{\omega}{\mathrm{k}} = \nu\lambda;\ \lambda = \frac{h}{p}$

$$p\mathbf{v}_p = h\nu = E = (c^2p^2 + m_0^2c^4)^{1/2}$$

$$p\mathbf{v}_p = cp\left(1+\frac{m_0^2c^4}{c^2p^2}\right)^{1/2} = cp\left(1+\frac{m_0^2c^2}{p^2}\right)^{1/2}$$

$$\mathbf{v}_p = c\left(1+\frac{m_0^2c^2}{p^2}\right)^{1/2} \quad \text{or } \mathbf{v}_p > c$$

$$\mathbf{v}_p = c\left(1+\frac{m_0^2c^2\lambda^2}{h^2}\right)^{1/2}$$

Hence $\mathbf{v}_p$ is a function of λ.

P/11 Show that the wavelength of a particle of rest mass m_0, with kintic energy T given by the relativistic formula $\lambda = \frac{hc}{\sqrt{T^2+2m_0c^2T}}$

Solution. ⇨ For a relativistic particle, we have$E^2 = c^2p^2 + m_0^2c^4$

Now, since $E = T + m_0c^2$

$$\text{Then, } (T + m_0c^2)^2 = c^2p^2 + m_0^2c^4$$

$$T^2 + 2m_0c^2T + m_0^2c^4 = c^2p^2 + m_0^2c^4$$

$$T^2 + 2m_0c^2T + m_0^2c^4 = c^2p^2 + m_0^2c^4$$

$$cp = \sqrt{T^2 + 2m_0c^2T}$$

$$\text{de Broglie wavelength } \lambda = \frac{h}{p} = \frac{hc}{\sqrt{T^2 + 2m_0c^2T}}$$

P/12 Find the condition at which de Broglie wavelength equals the Compton wavelength.

Solution. ⇨ Compton wavelength $\lambda_C = \frac{h}{m_0c}$

where m_0 is the rest mass of electron and c is the velocity of light

de Broglie wave length $\lambda = \frac{\mathrm{h}}{\mathrm{mv}}$

where m is the mass of electron$= \frac{m_0}{\sqrt{1-\mathbf{v}^2/c^2}}$ when its velocity is $\mathbf{v}$.

Then $\lambda = \frac{h\sqrt{1-\mathbf{v}^2/c^2}}{m_0\mathbf{v}} = \frac{h\sqrt{(c^2-\mathbf{v}^2)}}{m_0c\mathbf{v}} = \frac{h\mathbf{v}\sqrt{c^2/\mathbf{v}^2-1}}{m_0c\mathbf{v}} = \frac{h}{m_0c}\sqrt{c^2/\mathbf{v}^2 - 1} = \lambda_c\sqrt{\frac{c^2}{\mathbf{v}^2} - 1}$

When $\lambda = \lambda_C$

$$\sqrt{\frac{c^2}{\mathbf{v}^2} - 1} = 1 \text{ or } \frac{c^2}{\mathbf{v}^2} - 1 = 1$$

$$\frac{c^2}{\mathbf{v}^2} = 2 \text{ or } \mathbf{v} = \frac{c}{\sqrt{2}}$$

P/13 An electron has a de Broglie wavelength of 1.5×10^{-12} m. Find its (i) kinetic energy and (ii) group and phase velocities of its matter waves.

Solution. ⇨ (i) The total energy E of the electron is given by $E = \sqrt{c^2p^2 + m_0^2c^4}$

$$\text{Kinetic energy } T = E - m_0c^2 = \sqrt{c^2p^2 + m_0^2c^4} - m_0c^2$$

de Broglie wavelength $\lambda = \frac{h}{p}$ or $cp = \frac{hc}{\lambda}$

$$cp = \frac{(6.626 \times 10^{-34}\ \mathrm{J\ s})(3 \times 10^{8}\ \mathrm{ms^{-1}})}{1.5 \times 10^{-12}\ \mathrm{m}} = 13.252 \times 10^{-14}\ \mathrm{J}$$

$$\mathrm{E_0} = \mathrm{m_0 c^2} = (9.1 \times 10^{-31}\ \mathrm{kg})(3 \times 10^{8}\ \mathrm{m\ s^{-1}}) = 8.19 \times 10^{-14}\ \mathrm{J}$$

$T = \sqrt{(13.252)^2 + (8.19)^2} \times 10^{-14}\ \mathrm{J} - 8.19 \times 10^{-14}\ \mathrm{J} = 7.389 \times 10^{-14}\ \mathrm{J} = 4.62 \times 10^{5}\mathrm{eV}$

(ii) $E = \sqrt{(13.252)^2 + (8.19)^2} \times 10^{-14}\ \mathrm{J} = 15.579 \times 10^{-14}\ \mathrm{J}$

$$\mathrm{E} = \frac{E_0}{\sqrt{1 - v^2/c^2}} \text{ or } 1 - \frac{\mathbf{v}^2}{c^2} = \left(\frac{E_0}{E}\right)^2$$

$$v = \left[1 - \left(\frac{E_0}{E}\right)^2\right]^{1/2} c = \left[1 - \left(\frac{8.19}{15.579}\right)^2\right]^{1/2} (3 \times 10^{8}\ \mathrm{ms^{-1}}) = 0.851c$$

The group velocity will be the same as the particle velocity. Hence,

$$\mathbf{v}_g = 0.851c$$

$$\text{Phase velocity } \mathbf{v}_p = \frac{c^2}{\mathbf{v}} = \frac{c}{0.851} = 1.175\mathrm{c}$$

P/14 (i) The waves on the surface of water travel with a phase velocity $\mathbf{v}_p = \sqrt{g\lambda/2\pi}$, where g is the acceleration due to gravity and λ is the wavelength of the wave. Show that the group velocity of a wave packet comprised of these waves is $\mathbf{v}_p/2$.

(ii) For a relativistic particle, show that the velocity of the particle and the group velocity of the corresponding wave packet are the same.

Solution. ⇨ (i) The phase velocity $\mathbf{v}_p = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{g}{k}}$ where k is the wave vector.

By definition, $\mathbf{v}_p = \omega/k$, and hence $\frac{\omega}{\mathrm{k}} = \sqrt{\frac{\mathrm{g}}{\mathrm{k}}}$ or $\omega = \sqrt{\mathrm{gk}}$

The group velocity $\mathbf{v}_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{dk}} = \frac{1}{2}\sqrt{\frac{g}{\mathrm{k}}} = \frac{\mathrm{v_p}}{2}$

(ii) Group velocity $\mathbf{v}_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{dk}} = \frac{\mathrm{dE}}{\mathrm{dp}}$

For relativistic particle, $E^2 = c^2p^2 + m_0^2c^4$, and therefore,

$$\mathbf{v}_g = \frac{dE}{dp} = \frac{c^2p}{E} = \frac{c^2 m_0 \mathbf{v}\sqrt{1 - v^2/c^2}}{m_0 c^2 \sqrt{1 - v^2/c^2}} = v$$

P/15 If the total energy of a moving particle greatly exceeds its rest energy, show that its de Broglie wavelength is nearly the same as the wavelength of a photon with the same total energy.

Solution. ⇨ Let the total energy be E. Then,

$$E_2 = c^2p^2 + m_0^2c^4 \cong c^2p^2$$
$$p = \frac{E}{c}$$

de Broglie wavelength $\lambda = \frac{h}{p} = \frac{hc}{E}$ For a photon having the same energy,

$$E = hv = \frac{hc}{\lambda} \text{ or } \lambda = \frac{hc}{E}$$

which is the required result.

P/16 An electron microscope operates with a beam of electrons, each of which has an energy $60\mathrm{keV}$. W hat is the smallest size that such a device could resolve? What must be the energy of each neutron in a beam of neutrons be in order to resolve the same size of object?

Solution. ⇨The momentum of the electron is given by

$$p^2 = 2\mathrm{mE} = 2(9.1 \times 10^{-31}\ \mathrm{kg})(60 \times 1000 \times 1.6 \times 10^{-19}\ \mathrm{J})$$
$$p = 13.218 \times 10^{-23}\ \mathrm{kg\ ms^{-1}}$$

The de Broglie wavelength

$$\begin{aligned}\lambda &= \frac{h}{p} = \frac{6.626 \times 10^{-34}\mathrm{Js}}{13.216 \times 10^{-23}\ \mathrm{kg\ m\ s^{-1}}} \\ &= 5.01 \times 10^{-12}\ \mathrm{m}\end{aligned}$$

The smallest size an electron microscope can resolve is of the order of the de Broglie wavelength of electron. Hence the smallest size that can be resolved is $5.01 \times 10^{-12}\ \mathrm{m}$.

The de Broglie wavelength of the neutron must be of the order of $5.01 \times 10^{-12}\ \mathrm{m}$. Hence, the momentum of the neutron must be the same as that of electron. Then,

$$\text{Momentum of neutron } = 13.216 \times 10^{-23}\ \mathrm{kg\ m\ s^{-1}}$$

$$\begin{aligned}\text{Energy} &= \frac{\mathrm{p}^2}{2\mathrm{M}} \text{ (M is mass of neutron)} \\ &= \frac{(13.216 \times 10^{-23}\ \mathrm{kg\ ms^{-1}})^2}{2 \times 1836(9.1 \times 10^{-31}\ \mathrm{kg})} = 5.227 \times 10^{-18}\ \mathrm{J} \\ &= \frac{5.227 \times 10^{-18}\ \mathrm{J}}{1.6 \times 10^{-19}\ \mathrm{J/eV}} = 32.67\mathrm{eV}\end{aligned}$$

P/17 An electron has a speed of $500\ \mathrm{m/s}$ with an accuracy of 0.004%. Calculate the certainty with which we can locate the position of the electron.

Solution. ⇨ Momentum $p = mv = (9.11 \times 10^{-31}\ \mathrm{kg}) \times \left(\frac{500\ \mathrm{m}}{\mathrm{s}}\right)$

$$\frac{\Delta \mathrm{p}}{\mathrm{p}} \times 100 = 0.004$$

$$\Delta p = \frac{0.004(9.11 \times 10^{-31}\ \mathrm{kg})(500\ \mathrm{m/s})}{100}$$

$$= 182.2 \times 10^{-34}\ \mathrm{kg\ m\ s^{-1}}$$

$$\Delta \mathrm{x} \cong \frac{\mathrm{h}}{\Delta \mathrm{p}} = \frac{6.626 \times 10^{-34}\ \mathrm{J\ s}}{182.2 \times 10^{-34}\ \mathrm{kg\ ms^{-1}}} = 0.0364\ \mathrm{m}$$

The position of the electron cannot be measured to accuracy less than $0.036\ \mathrm{m}$.

P/18 The average lifetime of an excited atomic state is $10^{-9}\ \mathrm{s}$. If the spectral line associated with the decay of this state is 6000Å, estimate the width of the line.

Solution. ⇨ $\Delta t = 10^{-9}\ s,\ \lambda = 6000 \times 10^{-10}\ \mathrm{m} = 6 \times 10^{-7}\ \mathrm{m}$

$$\mathrm{E} = \frac{\mathrm{hc}}{\lambda} \text{ or } \Delta \mathrm{E} = \frac{\mathrm{hc}}{\lambda^2}\Delta\lambda$$

$$\Delta \mathrm{E} \cdot \Delta \mathrm{t} = \frac{\mathrm{hc}}{\lambda^2}\Delta\lambda \cdot \Delta \mathrm{t} \approx \frac{\hbar}{2} = \frac{\mathrm{h}}{4\pi}$$

$$\Delta\lambda = \frac{\lambda^2}{4\pi \mathrm{c} \Delta \mathrm{t}} = \frac{36 \times 10^{-14}\ \mathrm{m}^2}{4\pi(3 \times 10^8\ \mathrm{m/s}) \times (10^{-9}\ \mathrm{s})} = 9.5 \times 10^{-14}\ \mathrm{m}$$

P/19 An electron in the $n = 2$ state of hydrogen remains there on the average of about $10^{-8}\ s$, before making a transition to $n = 1$ state.

(i) Estimate the uncertainty in the energy of the $n = 2$ state.

(ii) What fraction of the transition energy is this?

(iii) What is the wavelength and width of this line in the spectrum of hydrogen atom?

Solution. ⇨ From Eq. (IV), (i) $\Delta \mathrm{E} \geq \frac{\mathrm{h}}{4\pi \Delta \mathrm{t}} = \frac{6.626 \times 10^{-34}\mathrm{Js}}{4\pi \times 10^{-8}\ \mathrm{s}}$

$$= 0.527 \times 10^{-26}\ \mathrm{J} = 3.29 \times 10^{-8}\mathrm{eV}$$

(ii) Energy of $n = 2 \rightarrow n = 1$ transition

$$= \quad -13.6\text{eV}\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.2\text{eV}$$

$$\text{Fraction } \frac{\Delta \text{E}}{\text{E}} = \frac{3.29 \times 10^{-8}\text{eV}}{10.2\text{eV}} = 3.23 \times 10^{-9}$$

(iii) $\lambda = \frac{hc}{E} = \frac{(6.626\times10^{-34}\ \text{J s})\times(3\times10^{8}\ \text{m/s})}{(10.2\times1.6\times10^{-19}\ \text{J})}$

$$= 1.218 \times 10^{-7}\ \text{m} = 122\ \text{nm}$$

$$\frac{\Delta \text{E}}{\text{E}} = \frac{\Delta\lambda}{\lambda} \text{ or } \Delta\lambda = \frac{\Delta \text{E}}{\text{E}} \times \lambda$$

$$\Delta\lambda = (3.23 \times 10^{-9})(1.218 \times 10^{-7}\ \text{m})$$

$$= 3.93 \times 10^{-16}\ \text{m} = 3.93 \times 10^{-7}\ \text{nm}$$

P/20 A subatomic particle produced in a nuclear collision is found to have a mass such that $Mc^2 = 1228\text{MeV}$, with an uncertainty of $\pm 56\text{MeV}$. E stimate the lifetime of this state. A ssuming that, when the particle is produced in the collision, it travels with a speed of $10^8\ \text{m/s}$, how far can it travel before it disintegrates?

Uncertainty in energy $\Delta\text{E} = (56 \times 10^6\text{eV})(1.6 \times 10^{-19}\ \text{J/eV})$

Solution. ⇨ $\Delta t \quad = \frac{\hbar}{2}\frac{1}{\Delta \text{E}} = \frac{(1.05\times10^{-34}\ \text{J s})}{2}\frac{1}{(56\times1.6\times10^{-13}\ \text{J})}$

$$= 5.86 \times 10^{-24}\ \text{s}$$

Its lifetime is about 5.86×10^{-24} s, which is in the laboratory frame.

Distance travelled before disintegration $= (5.86 \times 10 - 24\ \text{s})(10^8\ \text{m/s})$
$= 5.86 \times 10^{-16}\ \text{m}$

P/21 A bullet of mass 0.03 kg is moving with a velocity of $500\ \text{m s}^{-1}$. The speed is measured up to an accuracy of 0.02%. Calculate the uncertainty in x. Also comment on the result.

Solution. ⇨ Momentum $p = 0.03 \times 500 = 15\ \text{kg m s}^{-1}$

$$\frac{\Delta p}{p} \times 100 = 0.02$$

$$\Delta p = \frac{0.02 \times 15}{100} = 3 \times 10^{-3}\ \text{kg m s}$$

$$\Delta x \approx \frac{h}{2\Delta p} = \frac{6.626 \times 10^{-34}\text{Js}}{4\pi \times 3 \times 10^{-3}\ \text{km/s}} = 1.76 \times 10^{-31}\ \text{m}$$

As uncertainty in the position coordinate x is almost zero, it can be measured very accurately. In other words, the particle aspect is more predominant.

P/22 Wavelength can be determined with an accuracy of 1 in 10^8. What is the uncertainty in the position of a 10Å photon when its wavelength is simultaneously measured?

Solution. ⇨ $\Delta\lambda \approx 10^{-8}\text{ m},\ \lambda = 10 \times 10^{-10}\text{ m} = 10^{-9}\text{ m}$

$$p = \frac{h}{\lambda} \text{ or } \Delta p = \frac{h}{\lambda^2}\Delta\lambda$$
$$\Delta x \cdot \Delta p \cong \frac{\Delta x \times \Delta\lambda \times h}{\lambda^2}$$

From Eq. (iii), this product is equal to $\hbar/2$. Hence,

$$\frac{(\Delta x)(\Delta\lambda)h}{\lambda^2} = \frac{h}{4\pi}$$
$$\Delta x = \frac{\lambda^2}{4\pi\Delta\lambda} = \frac{10^{-18}\text{ m}^2}{4\pi \times 10^{-8}\text{ m}} = 7.95 \times 10^{-12}\text{ m}$$

P/23 If the position of a 5keV electron is located within 2Å, what is the percentage uncertainty in its momentum?

Solution. ⇨ $\Delta x = 2 \times 10^{-10}\text{ m};\ \Delta p \cdot \Delta x \cong \frac{h}{4\pi}$

$$\Delta p \cong \frac{h}{4\pi\Delta x} = \frac{(6.626 \times 10^{-34}\text{ J s})}{4\pi(2 \times 10^{-10}\text{ m})} = 2.635 \times 10^{-25}\text{ kg m s}^{-1}$$
$$p = \sqrt{2mT} = (2 \times 9.11 \times 10^{-31} \times 5000 \times 1.6 \times 10^{-19})^{1/2}$$
$$= 3.818 \times 10^{-23}\text{ kg m s}^{-1}$$

Percentage of uncertainty $= \frac{\Delta p}{p} \times 100 = \frac{2.635\times10^{-25}}{3.818\times10^{-23}} \times 100 = 0.69$

P/24 The uncertainty in the velocity of a particle is equal to its velocity. If $\Delta p \cdot \Delta x \cong h$, show that the uncertainty in its location is its de Broglie wavelength.

Solution. ⇨ Given $\Delta\mathbf{v} = \mathbf{v}$. Then,

$$\Delta p = m\Delta\mathbf{v} = m\mathbf{v} = p$$
$$\Delta x \times \Delta p \cong h \text{ or } \Delta x \cdot p \cong h$$
$$\Delta x \cong \frac{h}{p} = \lambda$$

P/25 An electron moves with a constant velocity $1.1 \times 10^6\text{ m/s}$. If the velocity is measured to a precision of 0.1 per cent, what is the maximum precision with which its position could be simultaneously measured?

Solution. The momentum of the electron is given by

$$
\begin{aligned}
p = &\quad (9.1 \times 10^{-31}\ \mathrm{kg})\left(1.1 \times \frac{10^{6}\ \mathrm{m}}{\mathrm{s}}\right) \\
= &\quad 1 \times \frac{10^{-24}\ \mathrm{kg\ m}}{\mathrm{s}} \\
&\quad \frac{\Delta \mathbf{v}}{\mathbf{v}} = \frac{\Delta p}{p} = \frac{0.1}{100} \\
\Delta p = &\quad p \times 10^{-3} = \frac{10^{-27}\ \mathrm{kg\ m}}{\mathrm{s}} \\
\Delta x \cong \frac{h}{4\pi \Delta \mathrm{p}} = &\quad \frac{6.626 \times 10^{-34} \mathrm{Js}}{4\pi \times \frac{10^{-27}\ \mathrm{kg\ m}}{\mathrm{s}}} = 6.6 \times 10^{-7}\ \mathrm{m}
\end{aligned}
$$

P/26 The position of an electron is measured with an accuracy of 10^{-6} m. Find the uncertainty in the electron's position after 1 s. Comment on the result.

Solution. ⇨ When $\mathrm{t} = 0$, the uncertainty in the electron's momentum is

Since $p = m\mathbf{v}, \Delta p = m\Delta \mathbf{v}$. Hence,

$$\Delta p \geq \frac{\hbar}{2\Delta x}$$

$$\Delta \mathbf{v} \geq \frac{\hbar}{2m\Delta x}$$

The uncertainty in the position of the electron at time t cannot be more than

$$
\begin{aligned}
(\Delta \mathrm{x})_{\mathrm{t}} &= \mathrm{t}\Delta \mathbf{v} \geq \frac{\hbar \mathrm{t}}{2\ \mathrm{m}\Delta \mathrm{x}} \\
&= \frac{(1.054 \times 10^{-34}\ \mathrm{J\ s})1\ \mathrm{s}}{2(9.1 \times 10^{-31}\ \mathrm{kg})10^{-6}\ \mathrm{m}} = 57.9\ \mathrm{m}
\end{aligned}
$$

The original wave packet has spread out to a much wider one. A large range of wave numbers must have been present to produce the narrow original wave group. The phase velocity of the component waves has varied with the wave number.

P/27 From scattering experiments, it is found that the nuclear diameter is of the order of 10^{-15} m. The energy of an electron in β-decay experiment is of the order of a few MeV. Use these data and the uncertainty principle to show that the electron is not a constituent of the nucleus.

Solution. ⇨If an electron exists inside the nucleus, the uncertainty in its position $\Delta \mathrm{x} \cong 10^{-15}$ m. From the uncertainty principle,

$$(10^{-15}\ \mathrm{m})\Delta \mathrm{p} \geq \frac{\hbar}{2}$$

$$\Delta \mathrm{p} \geq \frac{1.05 \times 10^{-34}\ \mathrm{J\ s}}{2(10^{-15}\ \mathrm{m})} = 5.25 \times 10^{-20}\ \mathrm{kg\ m\ s^{-1}}$$

The momentum of the electron p must at least be of this order.

$$p \cong 5.25 \times 10^{-20} \text{ kg m s}^{-1}$$

When the energy of the electron is very large compared to its rest energy,

$$\begin{aligned} \mathrm{E} \quad &\cong \mathrm{cp} = (3 \times 10^{8} \text{ ms}^{-1})(5.25 \times 10^{-20} \text{ kg m s}^{-1}) \\ &= \frac{15.75 \times 10^{-12} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 9.84 \times 10^{7} \text{eV} \\ &= 98.4 \text{MeV} \end{aligned}$$

This is very large compared to the energy of the electron in β-decay. Thus, electron is not a constituent of the nucleus.

P/28 What is the minimum energy needed for a photon to turn into an electron-positron pair? Calculate how long a virtual electron-positron pair can exist.

Solution. ⇨The M ass of an electron-positron pair is $2m_e c^2$. Hence the minimum energy needed to make an electron-positron pair is $2\ \mathrm{m_e c^2}$, i.e., this much of energy needs to be borrowed to make the electron-positron pair. By the uncertainty relation, the minimum time for which this can happen is

$$\begin{aligned} \Delta \mathrm{t} \quad &= \frac{\hbar}{2 \times 2\ \mathrm{m_e c^2}} \\ &= \frac{1.05 \times 10^{-34} \text{ J s}}{4(9.1 \times 10^{-31} \text{ kg})(3 \times 10^{8} \text{ m/s})^2} \\ &= 3.3 \times 10^{-22} \text{ s} \end{aligned}$$

which is the length of time for which such a pair exists.

P/29 A pair of virtual particles is created for a short time. During the time of their existence, a distance of $0.35\mathrm{fm}$ is covered with a speed very close to the speed of light. What is the value of $\mathrm{mc^2}$ (in eV) for each of the virtual particle?

Solution. ⇨According to Problem 2.43, the pair exists for a time Δ given by

$$\Delta \mathrm{t} = \frac{\hbar}{4\mathrm{mc^2}}$$

The time of existence is also given by

$$\Delta t = \frac{0.35 \times 10^{-15} \text{ m}}{3 \times 10^{8} \text{ m/s}} = 1.167 \times 10^{-24} \text{ s}$$

Equating the two expressions for Δt, we get

$$\frac{\hbar}{4mc^2} = 1.167 \times 10^{-24}\ \mathrm{s}$$

$$\begin{aligned} mc^2 &= \frac{1.05 \times 10^{-34}\mathrm{Js}}{4 \times 1.167 \times 10^{-24}\ \mathrm{s}} = 2.249 \times 10^{-11}\ \mathrm{J} \\ &= \frac{2.249 \times 10^{-11}\ \mathrm{J}}{1.6 \times 10^{-19}\ \mathrm{J/eV}} = 140.56 \times 10^{6}\mathrm{eV} \\ &= 140.56\mathrm{MeV} \end{aligned}$$

P/30 The uncertainty in energy of a state is responsible for the natural line width of spectral lines. Substantiate.

Solution. ⇨The equation $(\Delta E)(\Delta t) \geq \frac{\hbar}{2}$

implies that the energy of a state cannot be measured exactly unless an infinite amount of time is available for the measurement. If an atom is in an excited state, it does not remain there indefinitely, but makes a transition to a lower state. We can take the mean time for decay τ, called the lifetime, as a measure of the time available to determine the energy. Hence the uncertainty in time is of the order of τ. For transitions to the ground state, which has a definite energy E_0 because of its finite lifetime, the spread in wavelength can be calculated from

$$E - E_0 = \frac{hc}{\lambda}$$

$$|\Delta E| = \frac{hc|\Delta\lambda|}{\lambda^2}$$

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta E}{E - E_0}$$

Using Eq. (i) and identifying $\Delta t \cong \tau$, we get

$$\frac{\Delta\lambda}{\lambda} = \frac{\hbar}{2\tau(E - E_0)}$$

The energy width $\hbar/\tau$ is often referred to as the natural line width.

P/31 Consider the electron in the hydrogen atom. Using $(\Delta x)(\Delta p) \simeq \hbar$, show that the radius of the electron orbit in the ground state is equal to the Bohr radius.

Solution. ⇨The energy of the electron in the hydrogen atom is the given by

$$E = \frac{p^2}{2m} - \frac{ke^2}{r},\ k = \frac{1}{4\pi\varepsilon_0}$$

where p is the momentum of the electron. For the order of magnitude of the position uncertainty, if we take $\Delta x \cong r$, then

$$\Delta p \cong \frac{\hbar}{r} \text{ or } (\Delta p)^2 = \frac{\hbar^2}{r^2}$$

Taking the order of momentum p as equal to the uncertainty in momentum, we get

$$(\Delta \mathrm{p})^2 = \langle \mathrm{p}^2 \rangle = \frac{\hbar^2}{\mathrm{r}^2}$$

Hence, the total energy $E = \frac{\hbar}{2mr^2} - \frac{ke^2}{r}$

For E to be minimum, $(dE/dr) = 0$. Then,

$$\frac{\mathrm{dE}}{\mathrm{dr}} = -\frac{\hbar^2}{mr^3} + \frac{k\mathrm{e}^2}{\mathrm{r}^2} = 0$$

$$r = \frac{\hbar^2}{kme^3} = a_0$$

which is the required result.

P/32 Normalize the wave function $\psi(\mathrm{x}) = \mathrm{A}\exp(-ax^2), A$ and a are constants, over the domain $-\infty \le x \le \infty$.

Solution. ⇨ Taking A as the normalization constant, we get

$$\mathrm{A}^2 \int_{-\infty}^{\infty} \psi^* \psi \mathrm{dx} = \mathrm{A}^2 \int_{-\infty}^{\infty} \exp(-2a\mathrm{x}^2)\mathrm{dx} = 1$$

Using the result (see the A ppendix), we get

$$\int_{-\infty}^{\infty} \exp(-2ax^2)dx = \sqrt{\frac{\pi}{2a}}$$

$$A = \left(\frac{2a}{\pi}\right)^{1/4}$$

$$\psi(x) = \left(\frac{2a}{\pi}\right)^{1/4} \exp(-ax^2)$$

P/33 A particle constrained to move along the x-axis in the domain $0 \le x \le L$ has a wave function $\psi(\mathrm{x}) = \sin(\mathrm{n}\pi\mathrm{x}/\mathrm{L})$, where n is an integer. Normalize the wave function and evaluate the expectation value of its momentum.

Solution. ⇨ The normalization condition gives

$$N^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx = 1$$

$$N^2 \int_0^L \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right) dx = 1$$

$$N^2 \frac{L}{2} = 1 \text{ or } N = \sqrt{\frac{2}{L}}$$

The normalized wave function is $\sqrt{2/L}\sin[(n\pi x)/L]$.

So,

$$\begin{aligned}\langle p_x\rangle &= \int_0^L \psi^*\left(-i\hbar\frac{d}{dx}\right)\psi dx \\ &= -i\hbar\frac{2}{L}\frac{n\pi}{L}\int_0^L \sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}dx \\ &= -i\hbar\frac{n\pi}{L^2}\int_0^L \sin\frac{2n\pi x}{L}dx = 0\end{aligned}$$

P/34 Give the mathematical representation of a spherical wave travelling outward from a point and evaluate its probability current density.

Solution. ⇨ The mathematical representation of a spherical wave travelling outwards from a point is given by $\psi(r) = \frac{\mathrm{A}}{r}\exp(ikr)$

where A is a constant and k is the wave vector. The probability current density

$$\begin{aligned} j &= \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) \\ &= \frac{i\hbar}{2m}|A|^2\left[\frac{e^{ikr}}{r}\nabla\left(\frac{e^{-ikr}}{r}\right) - \frac{e^{-ikr}}{r}\nabla\left(\frac{e^{ikr}}{r}\right)\right] \\ &= \frac{i\hbar}{2m}|A|^2\left[\frac{e^{ikr}}{r}\left(-\frac{ik}{r}e^{-ikr} - \frac{e^{-ikr}}{r^2}\right) - \frac{e^{-ikr}}{r}\left(\frac{ik}{r}e^{ikr} - \frac{e^{ikr}}{r^2}\right)\right] \\ &= \frac{i\hbar}{2m}|A|^2\left(\frac{-2ik}{r^2}\right) = \frac{\hbar k}{mr^2}|A|^2\end{aligned}$$

P/35 The wave function of a particle of mass m moving in a potential $V(x)$ is $\Psi(x,t) = A\exp\left(-ikt - \frac{km}{\hbar}x^2\right)$, where A and k are constants. Find the explicit form of the potential $V(x)$.

Solution. ⇨

$$\begin{aligned}&\Psi(x,t) = A\exp\left(-ikt - \frac{kmx^2}{\hbar}\right) \\ &\frac{\partial\Psi}{\partial x} = -\frac{2kmx}{\hbar}\Psi \\ &\frac{\partial^2\Psi}{\partial x^2} = \left(-\frac{2km}{\hbar} + \frac{4k^2m^2x^2}{\hbar^2}\right)\Psi \\ &\mathrm{i}\hbar\frac{\partial\Psi}{\partial \mathrm{t}} = \mathrm{k}\hbar\Psi\end{aligned}$$

Substituting these values in the time dependent Schrödinger equation, we have

$$\begin{aligned}\mathrm{k}\hbar &= -\frac{\hbar^2}{2m}\left(-\frac{2\ \mathrm{km}}{\hbar} + \frac{4\mathrm{k}^2\ \mathrm{m}^2\mathrm{x}^2}{\hbar^2}\right) + \mathrm{V(x)} \\ \mathrm{k}\hbar &= \mathrm{k}\hbar - 2m\mathrm{k}^2\mathrm{x}^2 + \mathrm{V(x)} \\ \mathrm{V(x)} &= 2m\mathrm{k}^2\mathrm{x}^2\end{aligned}$$

P/36 The time-independent wave function of a system is $\psi(\mathrm{x}) = \mathrm{A}$ exp (ikx), where k is a constant. Check whether it is normalizable in the domain $-\infty < x < \infty$. Calculate the probability current density for this function.

Solution. ⇨ Substitution of $\psi(\mathrm{x})$ in the normalization condition gives

$$|\mathrm{N}|^2 \int_{-\infty}^{\infty} |\psi|^2 \mathrm{dx} = |\mathrm{N}|^2 \int_{-\infty}^{\infty} \mathrm{dx} = 1$$

As this integral is not finite, the given wave function is not normalizable in the usual sense. The probability current density

$$\begin{aligned} j &= \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) \\ &= \frac{i\hbar}{2m}|A|^2\left[e^{ikx}(-ik)e^{-ikx} - e^{-ikx}(ik)e^{ikx}\right] \\ &= \frac{i\hbar}{2m}|A|^2(-ik - ik) = \frac{k\hbar}{m}|A|^2 \end{aligned}$$

P/37 Calculate the probability current density $j(x)$ for the wave function.

$\psi(\mathrm{x}) = u(\mathrm{x})\exp[i\phi(\mathrm{x})]$, where u, ϕ are real.

Solution. ⇨

$$\begin{aligned} &\psi(\mathrm{x}) = u(\mathrm{x})\exp(\mathrm{i}\phi);\ \psi^*(\mathrm{x}) = u(\mathrm{x})\exp(-i\phi) \\ &\frac{\partial\psi}{\partial x} = \frac{\partial u}{\partial x}\exp(i\phi) + iu\frac{\partial\phi}{\partial x}\exp(i\phi) \\ &\frac{\partial\psi^*}{\partial x} = \frac{\partial u}{\partial x}\exp(-i\phi) - iu\frac{\partial\phi}{\partial x}\exp(i\phi) \\ &j(x) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right) \\ &= \frac{i\hbar}{2m}\left[ue^{i\phi}\left(\frac{\partial u}{\partial x}e^{-i\phi} - iu\frac{\partial\phi}{\partial x}e^{-i\phi}\right) - ue^{i\phi}\left(\frac{\partial u}{\partial x}e^{i\phi} + iu\frac{\partial\phi}{\partial x}e^{i\phi}\right)\right] \\ &= \frac{i\hbar}{2m}\left[u\frac{\partial u}{\partial x} - iu^2\frac{\partial\phi}{\partial x} - u\frac{\partial u}{\partial x} - iu^2\frac{\partial\phi}{\partial x}\right] \\ &= \frac{i\hbar}{2m}\left[-2iu^2\frac{\partial\phi}{\partial x}\right] = \frac{\hbar}{m}u^2\frac{\partial\phi}{\partial x} \end{aligned}$$

P/38 The time-independent wave function of a particle of mass m moving in a potential $V(x) = \alpha^2 x^2$ is $\psi(\mathrm{x}) = \exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\mathrm{x}^2\right)$, α being a constant.

Find the energy of the system.

Solution. ⇨ We have

$$\psi(x) = \exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}x^2\right)$$

$$\frac{d\psi}{dx} = -\sqrt{\frac{2m\alpha^2}{\hbar^2}} \times \exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}x^2\right)$$

$$\frac{d^2\psi}{dx^2} = -\sqrt{\frac{2m\alpha^2}{\hbar^2}}\left[1 - \sqrt{\frac{2m\alpha^2}{\hbar^2}}x^2\right]\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}x^2\right)$$

Substituting these in the time-independent Schrödinger equation and dropping the exponential term, we obtain

$$-\frac{\hbar^2}{2m}\left[-\sqrt{\frac{2m\alpha^2}{\hbar^2}} + \frac{2m\alpha^2}{\hbar^2}x^2\right] + a^2x^2 = E$$

$$\hbar\sqrt{\frac{\alpha^2}{2m}} - a^2x^2 + a^2x^2 = E$$

$$E = \frac{\hbar\alpha}{\sqrt{2m}}$$

P/39 For a particle of mass m, Schrödinger initially arrived at the wave equation

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial^2\Psi}{\partial x^2} - \frac{m^2c^2}{\hbar^2}\Psi$$

Show that a plane wave solution of this equation is consistent with the relativistic energy momentum relationship.

Solution. ⇨ For plane waves, $\Psi(x,t) = A\exp[i(kx - \omega t)]$

Substituting this solution in the given wave equation, we obtain

$$\begin{aligned}\frac{(-i\omega)^2}{c^2}\Psi &= (ik)^2\Psi - \frac{m^2c^2}{\hbar^2}\Psi \\ \frac{-\omega^2}{c^2} &= -k^2 - \frac{m^2c^2}{\hbar^2}\end{aligned}$$

Multiplying by $c^2\hbar^2$ and writing $\hbar\omega = \mathrm{E}$ and $\mathrm{k}\hbar = \mathrm{p}$, we get

$$E^2 = c^2p^2 + m^2c^4$$

which is the relativistic energy-momentum relationship.

P/40 Using the time-independent Schrödinger equation, find the potential $\mathrm{V(x)}$ and energy E for which the wave function

$$\psi(x) = \left(\frac{x}{x_0}\right)^n e^{-x/x_0}$$

where n, x_0 are constants, is an eigenfunction. A ssume that $V(x) \to 0$ as $x \to \infty$.

Solution. ⇨ Differentiating the wave function with respect to x, we get

$$\begin{aligned}
\frac{d\psi}{dx} &= \frac{n}{x_0}\left(\frac{x}{x_0}\right)^{n-1} e^{-x/x_0} - \frac{1}{x_0}\left(\frac{x}{x_0}\right)^n e^{-x/x_0} \\
\frac{d^2\psi}{dx^2} &= \frac{n(n-1)}{x_0^2}\left(\frac{x}{x_0}\right)^{n-2} e^{-x/x_0} - \frac{2n}{x_0^2}\left(\frac{x}{x_0}\right)^{n-1} e^{-x/x_0} + \frac{1}{x_0^2}\left(\frac{x}{x_0}\right)^n e^{-x/x_0} \\
&= \left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x} + \frac{1}{x_0^2}\right]\left(\frac{x}{x_0}\right)^n e^{-x/x_0} \\
&= \left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x} + \frac{1}{x_0^2}\right]\psi(x)
\end{aligned}$$

Substituting in the Schrödinger equation, we get

$$-\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x} + \frac{1}{x_0^2}\right]\psi + V\psi = E\psi$$

which gives the operator equation

$$E - V(x) = -\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x} + \frac{1}{x_0^2}\right]$$

When $\mathrm{x} \to \infty, \mathrm{V(x)} \to 0$. Hence,

$$\begin{aligned}
E &= -\frac{\hbar^2}{2mx_0^2} \\
V(x) &= \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x}\right]
\end{aligned}$$

P/41 Find that the form of the potential, for which $\psi(\mathrm{r})$ is constant, is a solution of the Schrödinger equation. What happens to probability current density in such a case?

Solution. ⇨ Since $\psi(\mathrm{r})$ is constant, $\nabla^2\psi = 0$.

Hence the Schrödinger equation reduces to $\mathrm{V}\psi = \mathrm{E}\psi$ or $\mathrm{V} = \mathrm{E}$

The potential is of the form V which is a constant. Since $\psi(r)$ is constant, $\nabla\psi = \nabla\psi^* = 0$. Consequently, the probability current density is zero.

P/42 Obtain the form of the equation of continuity for probability if the potential in the Schrödinger equation is of the form $V(r) = V_1(r) + iV_2(\boldsymbol{r})$, where V_1 and V_2 are real.

Solution. The probability density $P(\boldsymbol{r}, \mathrm{t}) = \Psi^*\Psi$. Then,

$$\mathrm{i}\hbar\frac{\partial \mathrm{P}}{\partial \mathrm{t}} = \mathrm{i}\hbar\frac{\partial}{\partial \mathrm{t}}(\Psi\Psi^*) = \Psi\left(\mathrm{i}\hbar\frac{\partial}{\partial \mathrm{t}}\Psi^*\right) + \Psi^*\left(\mathrm{i}\hbar\frac{\partial\Psi}{\partial \mathrm{t}}\right)$$

The Schrödinger equation with the given potential is given by

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial \mathrm{t}} = \frac{-\hbar^2}{2\ \mathrm{m}}\nabla^2\Psi + (\mathrm{V}_1 + \mathrm{iV}_2)\Psi$$

Substituting the values of $\mathrm{i}\hbar\frac{\partial\Psi}{\partial \mathrm{t}}$ and $\mathrm{i}\hbar\frac{\partial\Psi^*}{\partial \mathrm{t}}$, we have

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial \mathrm{P}}{\partial \mathrm{t}} &= \frac{\hbar^2}{2\ \mathrm{m}}(\Psi\nabla^2\Psi^* - \Psi^*\nabla^2\Psi) + 2\mathrm{i}_2\mathrm{P}\\
\mathrm{i}\hbar\frac{\partial \mathrm{P}}{\partial \mathrm{t}} &= \frac{\hbar^2}{2\ \mathrm{m}}[\nabla\cdot(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) + 2\mathrm{iV}_2\mathrm{P}]\\
\frac{\partial \mathrm{P}}{\partial \mathrm{t}} &= \nabla\cdot\left(-\frac{\mathrm{i}\hbar}{2\ \mathrm{m}}\right)(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) + 2\frac{\mathrm{V}_2}{\hbar}\mathrm{P}\\
\frac{\partial \mathrm{P}}{\partial \mathrm{t}} + \nabla\cdot\boldsymbol{j}(\boldsymbol{r}, \mathrm{t}) &= \frac{2\ \mathrm{V}_2}{\hbar}\mathrm{P}(\boldsymbol{r}, \mathrm{t})
\end{aligned}$$

P/43 For a one-dimensional wave function of the form $\Psi(x,t) = A\exp[i\phi(x,t)]$

show that the probability current density can be written as $\boldsymbol{j} = \frac{\hbar}{m}|A|^2\frac{\partial\phi}{\partial x}$

Solution. ⇨ The probability current density $\boldsymbol{j}(\boldsymbol{r}, \mathrm{t})$ is given by

$$\begin{aligned}
j(r,t) &= \frac{i\hbar}{2m}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)\\
\Psi(x,t) &= A\exp[i\phi(x,t)]\\
\Psi^*(x,t) &= A^*\exp[-i\phi(x,t)]\\
\nabla\Psi &= \frac{\partial\Psi}{\partial x} = iAe^{i\phi}\frac{\partial\phi}{\partial x}\\
\nabla\Psi^* &= \frac{\partial\Psi^*}{\partial x} = -iA^*e^{-i\phi}\frac{\partial\phi}{\partial x}
\end{aligned}$$

Substituting these values, we get

$$\begin{aligned}
\boldsymbol{j} &= \frac{i\hbar}{2m}\left[Ae^{i\phi}\left(-iA^*e^{-i\phi}\frac{\partial\phi}{\partial x}\right) - A^*e^{-i\phi}\left(iA^i\phi\frac{\partial\phi}{\partial x}\right)\right]\\
&= \frac{i\hbar}{2m}[-i|A|^2 - i|A|^2]\frac{\partial\phi}{\partial x} = \frac{\hbar}{m}|A|^2\frac{\partial\phi}{\partial x}
\end{aligned}$$

P/44 Let $\psi_0(\mathrm{x})$ and $\psi_1(\mathrm{x})$ be the normalized ground and first excited state energy eigenfunctions of a linear harmonic oscillator. At some instants of time, $A\psi_0 + B\psi_1$, where A and B are constants, is the wave function of the oscillator. Show that $\langle x \rangle$ is in general different from zero.

Solution. ⇨The normalization condition gives

$$\langle (\mathrm{A}\psi_0 + \mathrm{B}\psi_1) \mid (\mathrm{A}\psi_0 + \mathrm{B}\psi_1) \rangle = 1$$
$$\mathrm{A}^2\langle \psi_0 \mid \psi_0 \rangle + \mathrm{B}^2\langle \psi_1 \mid \psi_1 \rangle = 1 \text{ or } \mathrm{A}_2 + \mathrm{B}_2 = 1$$

Generally, the constants A and B are not zero. The average value of x is given by

$$\begin{aligned}\langle \mathrm{x} \rangle &= \langle (\mathrm{A}\psi_0 + \mathrm{B}\psi_1)|\mathrm{x}|(\mathrm{A}\psi_0 + \mathrm{B}\psi_1) \rangle \\ &= \mathrm{A}^2\langle \psi_0|\mathrm{x}|\psi_0 \rangle + \mathrm{B}^2\langle \psi_1|\mathrm{x}|\psi_1 \rangle + 2\mathrm{AB}\langle \psi_0|\mathrm{x}|\psi_1 \rangle\end{aligned}$$

since A and B are real and $\langle \psi_0|\mathrm{x}|\psi_1 \rangle = \langle \psi_1|\mathrm{x}|\psi_0 \rangle$. As the integrands involved is odd,

$$\langle \psi_0|\mathrm{x}|\psi_0 \rangle = \langle \psi_1|\mathrm{x}|\psi_1 \rangle = 0$$
$$\langle \mathrm{x} \rangle = 2\mathrm{AB}\langle \psi_0|\mathrm{x}|\psi_1 \rangle$$

which is not equal to zero.

P/45 Show that, if a particle is in a stationary state at a given time, it will always remain in a stationary state.

Solution. ⇨ Let the particle be in the stationary state $\Psi(x, 0)$ with energy E.

Then we have $\quad H\Psi(x, 0) = E\Psi(x, 0)$

where H is the Hamiltonian of the particle which is assumed to be real. At a later time, let the wave function be $\Psi(x, t)$, i.e.,

At time t,

$$\begin{aligned}\Psi(x, t) &= \Psi(x, 0)e^{-iEt/\hbar} \\ H\Psi(x, t) &= H\Psi(x, 0)e^{-iEt/\hbar} \\ &= E\Psi(x, 0)e^{-iEt/\hbar} \\ &= E\Psi(x, t)\end{aligned}$$

Thus, $\Psi(x, t)$ is a stationary state which is the required result.

P/46 The wave function of a one-dimensional system is

$\psi(x) = Ax^n e^{-x/a}$, where A, a, n are constant

If $\psi(\mathrm{x})$ is an eigenfunction of the Schrödinger equation, find the condition on $\mathrm{V}(\mathrm{x})$ for the energy eigenvalue $E = -\hbar^2/(2\mathrm{ma}^2)$. Also find the value of $V(x)$.

Solution. ⇨

$$\psi(x) = Ax^n e^{-x/a}$$
$$\frac{d\psi}{dx} = Anx^{n-1}e^{-x/a} - \frac{A}{a}x^n e^{-x/a}$$
$$\frac{d^2\psi}{dx^2} = Ae^{-x/a}\left[n(n-1)x^{n-2} - \frac{2n}{a}x^{n-1} + \frac{x^n}{a^2}\right]$$

With these values, the Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}Ae^{-x/a}\left[n(n-1)x^{n-2} - \frac{2n}{a}x^{n-1} + \frac{x^n}{a^2}\right] + V(x)Ax^n e^{-x/a} = EAx^n e^{-x/a}$$
$$-\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{ax} + \frac{1}{a^2}\right] = E - V(x)$$

From this equation, it is obvious that for the energy $E = -\hbar^2/2\mathrm{ma}^2, V(x)$ must tend to zero as $x \to \infty$. Then,

$$\begin{aligned} V(x) \quad &= -\frac{\hbar^2}{2ma^2} - \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{a} + \frac{1}{a^2}\right] \\ &= \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{ax}\right] \end{aligned}$$

P/47 Consider a particle described by the wave function $\Psi(\mathrm{x,t}) = \mathrm{e}^{i(\mathrm{kx}-\omega t)}$.

(i) Is this wave function an eigenfunction corresponding to any dynamical variable or variables? If so, name them.

(ii) Does this represent a ground state?

(iii) Obtain the probability current density of this function.

Solution. ⇨

(i) Allowing the momentum operator $-\mathrm{i}\hbar(d/dx)$ to operate on the function, we have

$$\begin{aligned} -i\hbar\frac{d}{dx}e^{i(kx-\omega t)} \quad &= i\hbar(ik)e^{i(kx-\omega t)} \\ &= \hbar k e^{i(kx-\omega t)} \end{aligned}$$

Hence, the given function is an eigenfunction of the momentum operator. Allowing the energy operator $-\mathrm{i}\hbar(\mathrm{d}/\mathrm{dt})$ to operate on the function, we have

$$\begin{aligned} i\hbar\frac{d}{dt}e^{i(kx-\omega t)} \quad &= i\hbar(-i\omega)e^{i(kx-\omega t)} \\ &= \hbar\omega e^{i(kx-\omega t)} \end{aligned}$$

Hence, the given function is also an eigenfunction of the energy operator with an eigenvalue $\hbar\omega$.

(ii) Energy of a bound state is negative. Here, the energy eigenvalue is $\hbar\omega$, which is positive. Hence, the function does not represent a bound state.

(iii) The probability current density

$$\begin{aligned} \boldsymbol{j} &= \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) \\ &= \frac{i\hbar}{2m}(-ik - ik) = \frac{\hbar k}{m} \end{aligned}$$

P/48 Show that the average kinetic energy of a particle of mass m with a wave function $\psi(\mathrm{x})$ can be written in the form $\mathrm{T} = \frac{\hbar^2}{2\,\mathrm{m}} \int_{-\infty}^{\infty} \left|\frac{\mathrm{d}\psi}{\mathrm{dx}}\right|^2 \mathrm{dx}$

Solution. ⇨ The average kinetic energy

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \psi^* \frac{d^2\psi}{dx^2} dx$$

Integrating by parts, we obtain

$$\langle T \rangle = -\frac{\hbar^2}{2\,\mathrm{m}} \left[\psi^* \frac{\mathrm{d}\psi}{\mathrm{dx}}\right]_{-\infty}^{\infty} + \frac{\hbar^2}{2\,\mathrm{m}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\psi^*}{\mathrm{dx}} \frac{\mathrm{d}\psi}{\mathrm{dx}} \mathrm{dx}$$

As the wave function and derivatives are finite, the integrated term vanishes, and so

$$\langle T \rangle = \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left|\frac{d\psi}{dx}\right|^2 dx$$

P/49 The energy eigenvalue and the corresponding Eigen function for a particle of mass m in a one-dimensional potential $V(x)$ are

$$E = 0,\ \psi(x) = \frac{A}{x^2 + a^2}$$

Deduce the potential $V(\mathrm{x})$.

Solution. ⇨ The Schrödinger equation for the particle with energy eigenvalue $E = 0$ is

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = 0,\ \psi = \frac{A}{x^2 + a^2} \\ \frac{d\psi}{dx^2} = -\frac{2Ax}{(x^2 + a^2)^2} \\ \frac{d^2\psi}{dx^2} = -2A\left[\frac{1}{(x^2 + a^2)^2} - \frac{4x^2}{(x^2 + a^2)^3}\right] \\ = \frac{2A(3x^2 - a^2)}{(x^2 + a^2)^3} \end{aligned}$$

Substituting the value of $d^2\psi/dx^2$, we get

$$-\frac{\hbar^2}{2m}\frac{2A(3x^2-a^2)}{(x^2+a^2)^3}+\frac{V(x)A}{x^2+a^2}=0$$
$$V(x)=\frac{\hbar^2(3x^2-a^2)}{m(x^2+a^2)^2}$$

P/50 Find the probability that a particle can be found between $x=0.45$ and 0.55 when the particle is limited to the x-axis and has wave forletron

$$\psi=\begin{cases}\text{ax} & \text{for } 0\le x\le 1\\ 0 & \text{for } x<0 \text{ and } x>1\end{cases}$$

Also find the average value of x

Solution. ⇨ do it yourself.