2.2 Waves and Optics

2.2.1 Waves and Optics (Theory)

Paper: PHS-A-CC-2-4-TH Credit:4

1. Oscillations 8 Lectures

   Differential equation of Simple Harmonic Oscillation and its solution. Kinetic energy, potential energy, total energy and their time average values. Damped oscillation. Forced oscillations: Transient and steady states; Resonance, sharpness of resonance; power dissipation and Quality Factor.
2. Superposition of Harmonic Oscillations 4 Lectures
   (a) Superposition of Two Collinear Harmonic oscillations having equal frequencies and different frequencies (Beats). (b) Superposition of Two Perpendicular Harmonic Oscillation for phase difference $\delta = 0, \pi/2, \pi$: Graphical and Analytical Methods, Lissajous Figures with equal and unequal frequency and their uses.
3. Wave motion 4 Lectures

   Plane and Spherical Waves. Longitudinal and Transverse Waves. Plane Progressive (Traveling) Waves. Wave Equation for travelling waves. Particle and Wave Velocities. (Solution of spherical wave equation may be assumed)
4. Superposition of Harmonic Waves 9 Lectures
   (a) Velocity of Transverse Vibrations of Stretched Strings, standing (Stationary) Waves in a String: Fixed and Free Ends. Analytical Treatment. Changes with respect to Position and Time. Energy of Vibrating String. Transfer of Energy. Normal Modes of Stretched Strings. (form of the solution of wave equation may be assumed). Plucked and Struck Strings. (b) Superposition of N Harmonic Waves. Phase and Group Velocities.
5. Wave optics 4 Lectures
   (a) Electromagnetic nature of light. Definition and properties of wave front. Huygens Principle. (b) Temporal and Spatial Coherence.
6. Interference 10 Lectures

   Division of amplitude and wave front. Young's double slit experiment. Lloyd's Mirror and Fresnel's Biprism. Phase change on reflection: Stokes' treatment. Interference in Thin Films: parallel and wedge shaped films. Fringes of equal inclination (Haidinger Fringes); Fringes of equal thickness (Fizeau Fringes). Newton's Rings: Measurement of wavelength and refractive index.
7. Interferometers 5 Lectures

   (a) Michelson Interferometer (1) Idea of form of fringes (No theory required), (2) Determination of Wavelength, (3) Wavelength Difference, (4) Refractive Index, and (5) Visibility of Fringes.
   (b)Multiple beam interferometry, Fabry-Perot interferometer.
8. Diffraction 16 Lectures
   (a) Fraunhofer diffraction: Single slit. Circular aperture (solution may be assumed), Resolving Power of a telescope. Double slit. Multiple slits. Diffraction grating. Resolving power of grating. Rayleigh criterion for resolution. (b) Fresnel Diffraction: Fresnel's Assumptions. Fresnel's Half-Period Zones for Plane Wave. Explanation of Rectilinear Propagation of Light. Theory of a Zone Plate: Multiple Foci of a Zone Plate. Fresnel's Integral, Fresnel diffraction pattern of a straight edge, a slit.

1. Oscillations 8 Lectures

Differential equation of Simple Harmonic Oscillation and its solution. Kinetic energy, potential energy, total energy and their time average values. Damped oscillation. Forced oscillations: Transient and steady states; Resonance, sharpness of resonance; power dissipation and Quality Factor.

Periodic motion

- If a particle moves such that it repeats its path regularly after equal intervals of time, it's motion is said to be periodic.
- The interval of time required to complete one cycle of motion is called time period of motion.
- If a body in periodic motion moves back and forth over the same path, then the motion is said to be vibratory or oscillatory.
- Examples of such motion are to and fro motion of pendulum, vibrations of a tuning fork, mass attached to a spring and many more.
- Every oscillatory motion is periodic but every periodic motion is not oscillatory for example motion of earth around the sun is periodic but not oscillatory.
- Simple Harmonic Motion (or SHM) is the simplest form of oscillatory motion.

In mechanics and physics, **simple harmonic motion** is a special type of periodic motion or oscillation where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

Simple harmonic motion is typified by the motion of a mass on a spring when it is subject to the linear elastic restoring force given by Hooke's law. The motion is sinusoidal in time and demonstrates a single resonant frequency.

Significance :

1. **Simple harmonic motion can serve as a mathematical model for a variety of motions, such as the oscillation of a spring.** In addition, other phenomena can be approximated by simple harmonic motion, including the motion of a simple pendulum as well as molecular vibration. For simple harmonic motion to be an accurate model for a pendulum, the net force on the object at the end of the pendulum must be proportional to the displacement. This is a good approximation when the angle of the swing is small.
2. **Simple harmonic motion provides a basis for the characterization of more complicated motions through the techniques of Fourier analysis.**

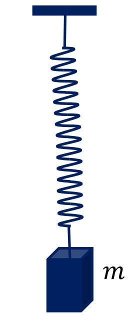
The motion of a particle moving along a straight line with an acceleration whose direction is always towards a fixed point on the line and whose magnitude is proportional to the distance from the fixed point is called simple harmonic motion [SHM]

For any simple mechanical harmonic oscillator: When the system is displaced from its equilibrium position, **a restoring force** that **obeys Hooke's law** tends to restore the system to equilibrium.

Once the mass is displaced from its equilibrium position, it experiences a net restoring force. As a result, it accelerates and starts going back to the equilibrium position. When the mass moves closer to the equilibrium position, the restoring force decreases. At the equilibrium position, the net restoring force vanishes. However, at $x = 0$, **the mass has momentum** because of the acceleration that the restoring force has imparted. Therefore, the mass continues past the equilibrium position, compressing the spring. A net restoring force then slows it down until its velocity reaches zero, whereupon it is accelerated back to the equilibrium position again.

As long as the system has no energy loss, the mass continues to oscillate. Thus **simple harmonic motion** is a type of **periodic motion.**

Note if the real space and phase space diagram **are not co-linear**, the phase space motion becomes elliptical. The area enclosed depends on the amplitude and the maximum momentum.



Real space

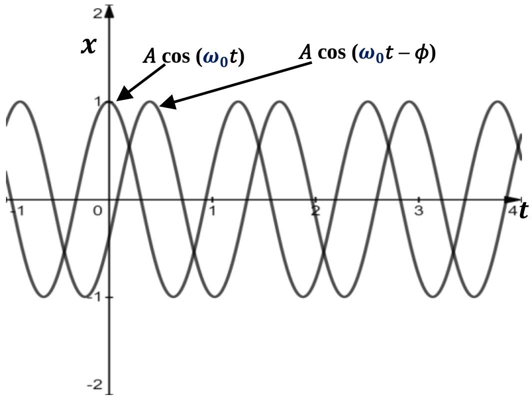
Differential equation of Simple Harmonic Oscillation

Suppose the mass of the particle is m, instantaneous displacement, velocity and acceleration are $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$

Then equation of motion of the said particle is $\boldsymbol{m}\frac{d^2x}{dt^2} = -s\,x$ **[Where s =restoring force per unit displacement]**

Or, $\boldsymbol{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$ **…………(i)** **[Where $\omega_0^2 = \frac{s}{m}$, ω_0 is called the natural frequency of the S.H.M]**

Its solution



Let $x = c\,e^{\alpha t}$ be a trial solution then from equation (i) we have

$c\,e^{\alpha t}(\alpha^2 + \omega_0^2) = 0$

As $c\,e^{\alpha t} \neq 0$ then $(\alpha^2 + \omega_0^2) = 0$ then $\alpha = \pm i\boldsymbol{\omega_0}$

Then the general solution is $x = x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$

Hence for a particle executing SHM the instantaneous displacement can be

written as $x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} = \{(c_1 + c_2)\cos\boldsymbol{\omega_0} t + i\,(c_1 - c_2)\sin\boldsymbol{\omega_0} t\}$

$= \{A\cos\phi\cos\boldsymbol{\omega_0} t + A\sin\phi\sin\boldsymbol{\omega_0} t\}$ **[Where $(c_1 + c_2) = A\cos\phi$ and $i(c_1 - c_2) = A\sin\phi$]**

Hence $\boldsymbol{x = x(t) = A\cos(\omega_0 t - \phi)}$ [here ϕ is known as initial phase or epoch]

Kinetic energy

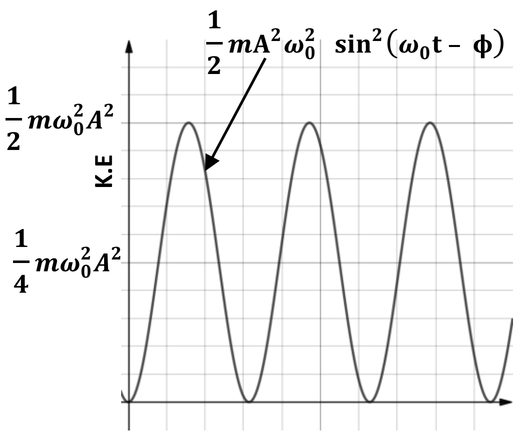
For any particle of mass m executing S.H.M the instantaneous displacement is given by $x(t) = A\cos(\omega_0 t - \phi)$.

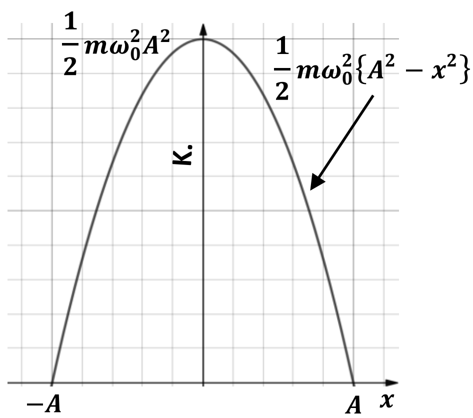
Then velocity of the system $\frac{dx}{dt} = -A\omega_0\sin(\omega_0 t - \phi) = A\omega\cos(\omega_0 t - \phi + \frac{\pi}{2})$.

[Hence velocity the lead to displacement by a phase amount $\frac{\pi}{2}$.]

Kinetic energy of the system $= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}mA^2\boldsymbol{\omega_0^2}\sin^2(\omega_0 t - \phi) = f(t)$

$$= \frac{1}{2}mA^2\boldsymbol{\omega_0^2}\,[1 - \cos^2(\omega_0 t - \phi)] = \frac{1}{2}mA^2\boldsymbol{\omega_0^2}\left(1 - \frac{x^2}{A^2}\right) = f(x)$$





Average Kinetic energy of the system

$$= \langle\tfrac{1}{2}mA^2\boldsymbol{\omega_0^2}\sin^2(\omega_0 t - \phi)\rangle = \tfrac{1}{2}mA^2\boldsymbol{\omega_0^2}\langle\sin^2(\omega_0 t - \phi)\rangle = \tfrac{1}{2}mA^2\boldsymbol{\omega_0^2}.\tfrac{1}{2} = \tfrac{1}{4}mA^2\boldsymbol{\omega_0^2}$$

Maximum Kinetic energy of the system $= \frac{1}{2}mA^2\boldsymbol{\omega_0^2}\sin^2(\omega_0 t - \phi)\Big|_{max} = \frac{1}{2}mA^2\boldsymbol{\omega_0^2}$

$$\boxed{\text{Maximum Kinetic energy} = 2\times\text{Average Kinetic energy}}$$

Since $\sin^2(\omega_0 t - \phi) = \frac{1}{2}[1 - \cos(2\omega_0 t - \phi)]$ we can say that instantaneous kinetic energy oscillates with the frequency $2\omega_0$

Potential energy

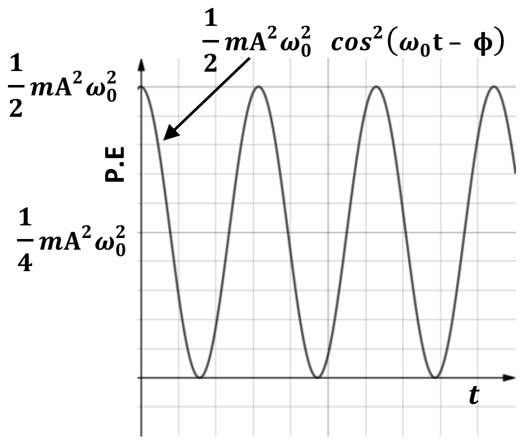
Here the restoring force is $-sx$. Hence work done against restoring force due to infinitesimal displacement dx is $-(-sx)dx$

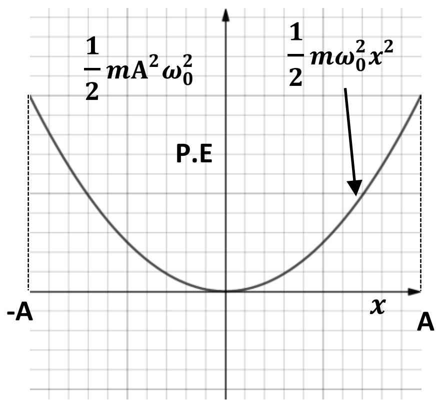
Hence total work done due to total displacement x from equilibrium point is $\int_0^x sx\,dx = \frac{1}{2} s\,x^2$.

This work done is stored as potential energy. Hence instantaneous potential energy P.E$= \frac{1}{2} s\,A^2 \cos^2(\omega_0 t - \phi)$

Hence potential energy of the system $= \frac{1}{2} m A^2 \omega_0^2 \ \cos^2(\omega_0 t - \phi) \ = f(t)$

$$= \frac{1}{2} m\omega_0^2\, x^2 = f(x)$$





Average Kinetic energy of the system

$= \langle \frac{1}{2} m A^2 \omega_0^2 \ \cos^2(\omega_0 t - \phi) \rangle = \frac{1}{2} m A^2 \omega_0^2 \ \langle \cos^2(\omega_0 t - \phi) \rangle = \frac{1}{2} m A^2 \omega_0^2 . \frac{1}{2} = \frac{1}{4} m A^2 \omega_0^2$

Maximum Kinetic energy of the system $= \frac{1}{2} m A^2 \omega_0^2 \ \cos^2(\omega_0 t - \phi)\Big|_{max} = \frac{1}{2} m A^2 \omega_0^2$

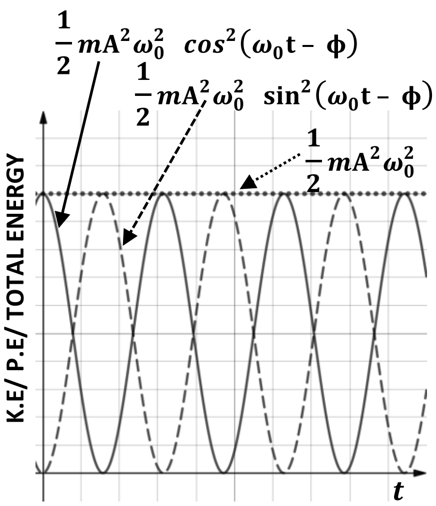
$\boxed{\text{Maximum potential energy} = 2 \times \text{Average potential energy}}$

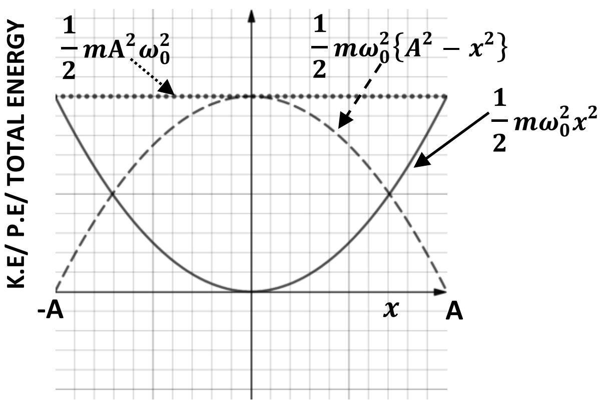
Since $\cos^2(\omega_0 t - \phi) = \frac{1}{2}[1 + \sin(2\omega_0 t - \phi)]$ we can say that instantaneous potential energy oscillates with the frequency $2\omega_0$

Total energy

Total energy, $E = \frac{1}{2} m A^2 \omega_0^2 \ \sin^2(\omega_0 t - \phi) + \frac{1}{2} m A^2 \omega_0^2 \ \cos^2(\omega_0 t - \phi) = \frac{1}{2} m A^2 \omega_0^2 \ =$ constant

Or, Total energy, $E = \frac{1}{2} m A^2 \omega_0^2 \ \left(1 - \frac{x^2}{A^2}\right) + \frac{1}{2} m\omega_0^2\, x^2 = \frac{1}{2} m A^2 \omega_0^2 \ =$ constant





Total energy= Maximum kinetic energy = Maximum potential energy

&

Note that: Average kinetic energy = Average potential energy

Differential equation of SHM from energy consideration

We have the total energy as $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\frac{1}{2}s\,x^2=E=constant$

Differentiating both site with respect to t we get $\frac{1}{2}m\;2\;\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{1}{2}s\;2\;x\;\frac{dx}{dt}=0$

As instantaneous $\frac{dx}{dt}\neq 0$ then $m\frac{d^2x}{dt^2}+s\;x\;=0$

Damped oscillation

when there are no dissipative forces the body set into oscillation will continue it for ever with constant amplitude, such an ideal vibration is called free vibration. It is characteristic by its natural frequency.

However, in a real physical system there is always some dissipative forces with all this the system to loss energy with time. It always results in a damping of the oscillation. Such forces are called damping forces. The damping force in a real system arise due to a number of complex physical process. When a body oscillates in a gas or a liquid damping forces arise due to the viscosity. When the oscillating body is always in contact with the solid surface the damping force is of fictional nature, this is termed Coulomb damping force. In some oscillating solid bodies, the damping force may arise due to imperfect elasticity of internal friction of the material, this is termed as structural damping. An accelerated charged particle emits electromagnetic radiation as a result of it a damping force arises, it is termed as radiation damping. **Note that radiation damping is proportional to acceleration.**

The viscous damping is easier to handle mathematically. For this it is a common practice to approximate the damping in a system by an equivalent viscous damping which is a function of velocity.

Let us consider the motion of a simple harmonic oscillators in the presence of damping forces. As said earlier the damping force is proportional to the velocity.

Differential equation of damped vibration

suppose the mass of the particle is m, instantaneous displacement, velocity and acceleration are $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$

Then equation of motion of the said particle is $\boldsymbol{m}\frac{d^2x}{dt^2} = -s\,x - \mu\frac{dx}{dt}$

[Where s =restoring force per unit displacement and μ =damping force per unit velocity]

Or, $\boldsymbol{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0}$ **…………(i)**

[Where $\omega_0^2 = \frac{s}{m}$, ω_0 is called the natural frequency of the S.H.M and $\frac{\mu}{m} = 2b$, b is called damping factor]

Its solution

Let $x = c\,e^{\alpha t}$ be a trial solution then $c\,e^{\alpha t}(\alpha^2 + 2b\alpha + \omega_0^2) = 0$

As $c\,e^{\alpha t} \neq 0$ then $(\alpha^2 + 2b\alpha + \omega_0^2) = 0$ then $\alpha = -b \pm \sqrt{b^2 - \omega_0^2}$

Then the general solution is $x = x(t) = C_1 e^{\left(-b+\sqrt{b^2-\omega_0^2}\right)t} + C_2 e^{\left(-b-\sqrt{b^2-\omega_0^2}\right)t} = e^{-bt}\left(C_1 e^{\sqrt{b^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{b^2-\omega_0^2}\,t}\right)$

<u>Case 1: for large damping for large damping $b > \omega_0$</u>

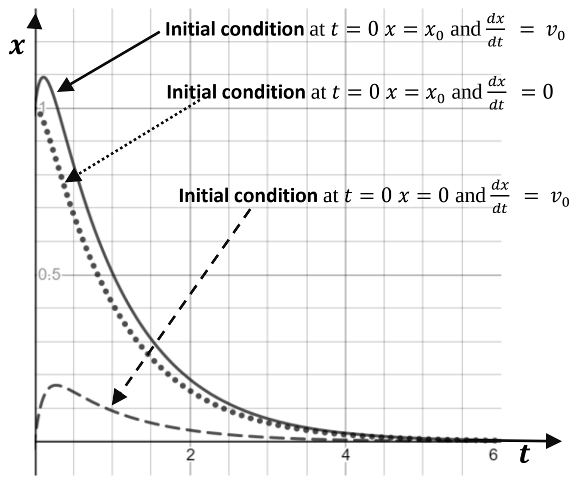
then $b^2 - \omega_0^2 > 0$ **hence the solution is non-oscillators.**

And given by $x(t) = e^{-bt}\left(C_1 e^{\sqrt{b^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{b^2-\omega_0^2}\,t}\right)$

However, the actual behaviour depends upon initial condition.

Let the initial conditions are at $t = 0$ $x = x_0$ and $\frac{dx}{dt} = v_0$

then if we take $-b + \sqrt{b^2 - \omega_0^2} = \alpha_1$ & $-b - \sqrt{b^2 - \omega_0^2} = \alpha_2$ then



$C_1 + C_2 = x_0$ & $\alpha_1 C_1 + \alpha_2 C_2 = v_0$ then $C_1 = \frac{x_0\alpha_2 - v_0}{\alpha_2 - \alpha_1} = \frac{x_0}{2}\left(1 + \frac{b + \frac{v_0}{x_0}}{\sqrt{b^2-\omega_0^2}}\right)$ & $C_1 = \frac{v_0 - x_0\alpha_1}{\alpha_2 - \alpha_1} = \frac{x_0}{2}\left(1 - \frac{b + \frac{v_0}{x_0}}{\sqrt{b^2-\omega_0^2}}\right)$

Hence $\boldsymbol{x(t) = \frac{x_0}{2} e^{-bt}\left\{\left(1 + \frac{b + \frac{v_0}{x_0}}{\sqrt{b^2-\omega_0^2}}\right) e^{\sqrt{b^2-\omega_0^2}\,t} + \left(1 - \frac{b + \frac{v_0}{x_0}}{\sqrt{b^2-\omega_0^2}}\right) e^{-\sqrt{b^2-\omega_0^2}\,t}\right\}}$

If the initial conditions are at $t = 0$ $x = x_0$ and $\frac{dx}{dt} = 0$

Then solution becomes $\boldsymbol{x(t)} = \frac{x_0}{2} \boldsymbol{e^{-bt}} \left\{ \left(\boldsymbol{1} + \frac{b}{\sqrt{b^2-\omega_0^2}} \right) \boldsymbol{e}^{\sqrt{b^2-\omega_0^2}\,t} + \left(\boldsymbol{1} - \frac{b}{\sqrt{b^2-\omega_0^2}} \right) \boldsymbol{e}^{-\sqrt{b^2-\omega_0^2}\,t} \right\}$

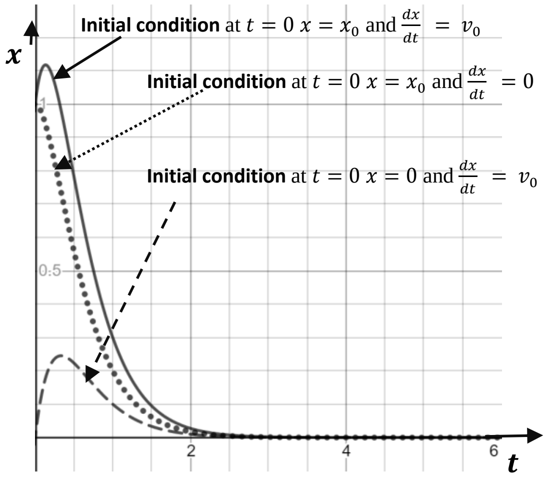
If the initial conditions are at $t = 0$ $x = 0$ and $\frac{dx}{dt} = v_0$

Then solution becomes $\boldsymbol{x(t)} = \left(\frac{v_0}{2\sqrt{b^2-\omega_0^2}} \right) \boldsymbol{e^{-bt}} \left\{ \boldsymbol{e}^{\sqrt{b^2-\omega_0^2}\,t} + \boldsymbol{e}^{-\sqrt{b^2-\omega_0^2}\,t} \right\} = \frac{v_0}{\sqrt{b^2-\omega_0^2}} \boldsymbol{e^{-bt}}\ \mathbf{sinh} \sqrt{\boldsymbol{b^2 - \omega_0^2}}\ \boldsymbol{t}$

Case 2: for critical damping for critical damping $b = \omega_0$

Then $\alpha_1 = \alpha_2 = -b\ or, -\omega_0,$ hence two roots are equal hence general solution of damped vibration in this case becomes

$x = (C_1 + C_2 t)e^{-bt}\ \ or, (C_1 + C_2 t)e^{-\omega_0 t}$



Let the initial conditions are at $\boldsymbol{t = 0}$ $\boldsymbol{x = x_0}$ **and** $\boldsymbol{\frac{dx}{dt} = v_0}$

Then $C_1 = x_0$ and $v_0 = -\omega_0 C_1 + C_2$ then $C_2 = v_0 + \omega_0 x_0$

$\boldsymbol{x = \{x_0 + (v_0 + \omega_0 x_0)t\}e^{-\omega_0 t}}$

If the initial conditions are at $\mathbf{t = 0}$ $\mathbf{x = x_0}$ **and** $\mathbf{\frac{dx}{dt} = 0}$

$\boldsymbol{x = x_0(1 + \omega_0 t)e^{-\omega_0 t}}$

If the initial conditions are at $\boldsymbol{t = 0}$ $\boldsymbol{x = 0}$ **and** $\boldsymbol{\frac{dx}{dt} = v_0}$

$\boldsymbol{x = \{v_0 t\}e^{-\omega_0 t}}$

Case 3: for under damping for under damping $b < \omega_0$

Then $\sqrt{b^2 - \omega_0^2} = \sqrt{-(\omega_0^2 - b^2)} = \sqrt{-\omega^2}$ say

Hence for underdamped motion $x = e^{-bt}\,(c_1 e^{i\omega t} + c_2 e^{-i\omega t}) = e^{-bt}\,\{(c_1 + c_2)\cos\omega t + i\,(c_1 - c_2)\sin\omega t\}$

$= e^{-bt}\,\{A\cos\phi\cos\omega t + A\sin\phi\sin\omega t\}$ **[Where $(c_1 + c_2) = A\cos\phi$**

$= A\,e^{-bt}\cos(\omega t - \phi)$ **$i(c_1 - c_2) = A\sin\phi$]**

Hence under damped motion is a damped oscillatory motion with frequency $\omega = \sqrt{(\omega_0^2 - b^2)}$ which is slightly less than natural frequency.

Let the initial conditions are at $\boldsymbol{t = 0}$ $\boldsymbol{x = x_0}$ **and** $\boldsymbol{\frac{dx}{dt} = v_0}$

Then $x_0 = A\cos\phi$ and $v_0 = -Ab\cos\phi + A\omega\sin\phi$ or, $v_0 = -\,bx_0 + A\omega\sin\phi$ or, $A\sin\phi = \frac{(v_0 + bx_0)}{\omega}$

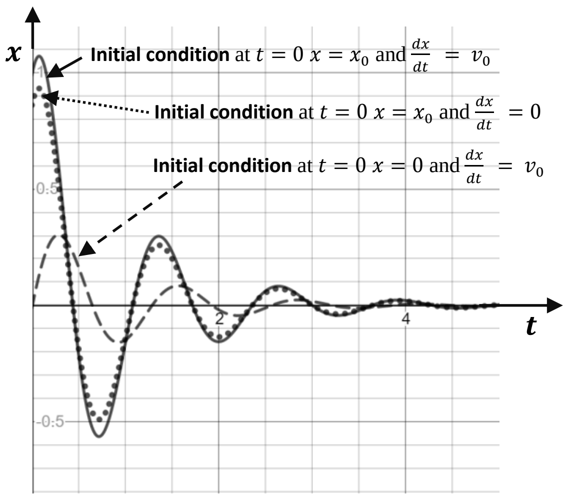
Then $A = \sqrt{x_0{}^2 + \frac{(v_0 + bx_0)^2}{\omega^2}}$ and $\tan\phi = \frac{(v_0 + bx_0)}{x_0\omega}$ or, $\phi = \tan^{-1}\frac{(v_0 + bx_0)}{x_0\omega}$

Hence $\boldsymbol{x} = \sqrt{\boldsymbol{x_0{}^2} + \frac{(v_0 + bx_0)^2}{\omega^2}}\ \boldsymbol{e^{-bt}}\ \mathbf{cos}\ \boldsymbol{(\omega t - \tan^{-1}\frac{(v_0 + bx_0)}{x_0\omega})}$

If the initial conditions are at $\mathbf{t = 0}$ $\mathbf{x = x_0}$ **and** $\mathbf{\frac{dx}{dt} = 0}$

$x = \sqrt{\frac{x_0{}^2((\omega_0^2 - b^2)) + (bx_0)^2}{\omega^2}}\ e^{-bt}\cos(\omega t - \tan^{-1}\frac{(b)}{\omega})$

$= \frac{x_0\omega_0}{\omega}\boldsymbol{e^{-bt}}\ \mathbf{cos}\ \boldsymbol{(\omega t - \tan^{-1}\frac{b}{\omega})}$



If the initial conditions are at $\boldsymbol{t = 0}$ $\boldsymbol{x = 0}$ **and** $\boldsymbol{\frac{dx}{dt} = v_0}$

Hence $\boldsymbol{x} = \frac{v_0}{\omega}\boldsymbol{e^{-bt}}\ \mathbf{cos}\ \boldsymbol{(\omega t - \tan^{-1}\infty)} = \frac{v_0}{\omega}\boldsymbol{e^{-bt}}\ \mathbf{sin}\ \boldsymbol{(\omega t)}$