In the given section we consider moving bodies with variable mass. Such kind of motion often occurs in nature and technology. We can mention here for example:

- Falling of an evaporating raindrop;
- Movement of a melting ice block or an iceberg on the ocean surface;
- Movement of a squid or jellyfish;
- Rocket flight.

**<u>Differential Equation of Rocket Motion in free space where there is no external force like gravity.</u>**

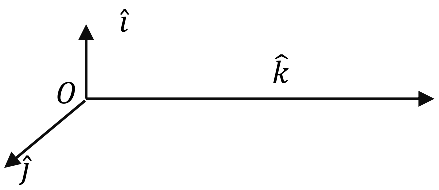
Rocket motion is based on Newton's third law, which states that "for every action there is an equal and opposite reaction".

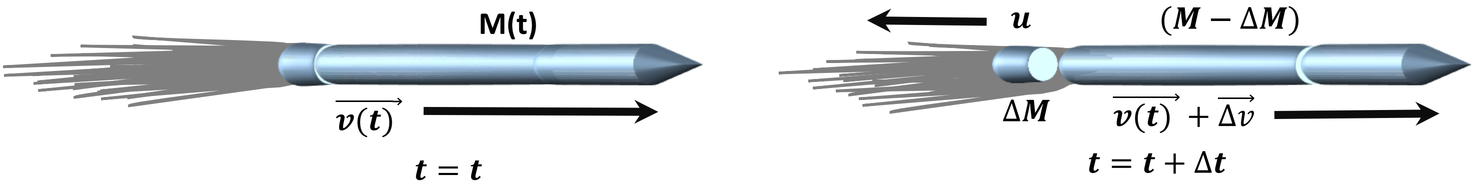
**Hot gases are exhausted through a nozzle of the rocket and produce the action force.**

The **reaction force acting in the opposite direction is called the thrust force.** The thrust force just causes the rocket acceleration.

Let consider the rocket is moving along + ve Z direction in a space where there is **no external force like gravity.**

Let at any time $t$ the mass of the rocket be $M(t)$ and its velocity be $\overrightarrow{v(t)} = v(t)\,\hat{k}$ w.r.t any fixed co-ordinate system. In certain time $\Delta t$ , the mass of the rocket decreases by $\Delta M$ as a result of the fuel combustion. This leads the rocket velocity to be increased by $\overrightarrow{\Delta v} = \Delta v(t)\,\hat{k}$ . And as a result of the fuel combustion let $\vec{u} = u(-\hat{k})$ is the exhaust gas velocity w.r.t the rocket.




Applying the **law of conservation of total momentum** to the system of the rocket and gas flow along Z axis we get.

$Mv = (M - \Delta M)(v + \Delta v) - \Delta M\,(u - v)$ [ **where $(u - v)$ is the exhaust gas velocity w.r.t any fixed co-ordinate system** ]

Or, $\cancel{Mv} = \cancel{Mv} + M\Delta v - \cancel{\Delta M v} - \Delta M \Delta v - \Delta M u + \cancel{\Delta M v}$

Or, $M\Delta v = u\,\Delta M$ **[ neglecting $\Delta M \Delta v$]**

Or, $\lim_{\Delta t \to 0} \frac{M\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{u\Delta M}{\Delta t}$

Or, $M\frac{dv}{dt} = -u\,\frac{dM}{dt}$ **[as mass of the rocket is decreasing we take $\lim_{\Delta t \to 0} \frac{\Delta M}{\Delta t} = -\frac{dM}{dt}$ ]**

Or, $M\,dv = -u\,dM$ or, $\boxed{dv = -u\frac{dM}{M}}$

Now if we consider that initial (at $t = 0$) mass and magnitude of velocity of the rocket are $M_0$ & $v_0$ then

$\int_{v_0}^{v(t)} dv = -u\,\int_{M_0}^{M(t)} \frac{dM}{M}$

Or, $v(t) - v_0 = -u\,\ln\frac{M(t)}{M_0}$ or, $\boxed{v(t) = v_0 - u\,\ln\frac{M(t)}{M_0}}$

**NOTE THAT: -**

1. If we consider that fuel is burning at a uniform rate, in that case the mass of the rocket will decrease uniformly. Let us consider the rate of **change** of the mass of rocket that means $\frac{dM}{dt} = \alpha$, **which is also known as the rate of consumption of fuel.**

   **then** $M(t) = M_0 - \alpha t$ **Then** $v(t) = v_0 - u \ln\frac{M(t)}{M_0} = v_0 - u \ln\frac{M_0-\alpha t}{M_0}$……………..(i)

2. If the mass of the empty rocket be $M'$ then total initial fuel in a rocket is $(M_0 - M')$. As long as there is fuel, the rocket will continue to accelerate but at that moment the fuel will burn completely there is no **action force or thrust force** hence there will be no acceleration. From then the rocket will travel at a uniform velocity.

3. Now if total time of acceleration of the rocket is T then $\frac{dM}{dt} T = \alpha T = (M_0 - M')$ **or,** $\boxed{T = \frac{(M_0-M')}{\alpha}}$

4. **Then final velocity of the rocket is** $v_F = v(T) = v_0 - u \ln\frac{M(t)}{M_0} = v_0 - u \ln\frac{M_0-\alpha T}{M_0}$

   **Or,** $\boxed{v_F = v_0 - u \ln\frac{M'}{M_0}}$ [ As $M_0 - \alpha T = M'$ ]

5. Now from the equation (i) $v(t) = \frac{dx(t)}{dt} = v_0 - u \ln\frac{M_0-\alpha t}{M_0}$

$$\text{Or, } \int_{x_0}^{x(t)} dx = \int_0^t v_0\, dt - u\int_0^t \ln\frac{M_0-\alpha t}{M_0}\, dt$$

$$\text{Or, } x(t) - x_0 = v_0 t - u\int_0^t \ln\frac{M_0-\alpha t}{M_0}\, dt \qquad \text{let } \frac{M_0-\alpha t}{M_0} = p \text{ then } -\frac{\alpha}{M_0}dt = dp$$

| $t$ | $0$ | $t$ |
|---|---|---|
| $p$ | $1$ | $\frac{M_0-\alpha t}{M_0}$ |

$$\text{Or, } x(t) - x_0 = v_0 t - u\left(-\frac{M_0}{\alpha}\right)\int_1^{\frac{M_0-\alpha t}{M_0}} \ln p\ dp$$

$$\text{Or, } x(t) - x_0 = v_0 t + \frac{u\,M_0}{\alpha}\left[p\ \ln p - p\right]_1^{\frac{M_0-\alpha t}{M_0}}$$

$$\text{Or, } x(t) = x_0 + v_0 t + \frac{u\,M_0}{\alpha}\left[\left(\frac{M_0-\alpha t}{M_0}\ln\frac{M_0-\alpha t}{M_0} - \frac{M_0-\alpha t}{M_0}\right) - (1\ \ln 1 - 1)\right]$$

$$\text{Or, } x(t) = x_0 + v_0 t + \frac{u\,M_0}{\alpha}\left[\left(\frac{M_0-\alpha t}{M_0}\ln\frac{M_0-\alpha t}{M_0} - 1 + \frac{\alpha t}{M_0} + 1\right)\right]$$

$$\text{Or, } x(t) = x_0 + v_0 t + u\left[\left(\frac{M_0-\alpha t}{\alpha}\ln\frac{M_0-\alpha t}{M_0} + t\right)\right]$$

So until the fuel burns off the total distance covered $x_F = x(T) = x_0 + v_0 T + u\left[\left(\frac{M_0-\alpha T}{\alpha}\ln\frac{M_0-\alpha t}{M_0} + T\right)\right]$

$$= x_0 + v_0 T + u\left[\frac{M'}{\alpha}\ln\frac{M'}{M_0} + \frac{(M_0-M')}{\alpha}\right]$$

6. Hence if there is **no external force like gravity the differential E.O.M of the rocket is** $M\frac{dv}{dt} = -u\frac{dM}{dt}$ **where M is the mass of the rocket,** $\frac{dv}{dt}$ **is that acceleration of the rocket,** $u$ is the **exhaust gas velocity** w.r.t the rocket. $\frac{dM}{dt} = \alpha$ **is the rate of consumption of fuel.**

**<u>The differential equation of motion of any rocket under gravity (vertical ascent)</u>**

Let us assume that the rocket has only a vertical motion, and there is no air resistance, and the gravitational field is constant with height. Adding the gravitational contribution we can write the differential equation of motion of the rocket as $\boxed{\boldsymbol{M\frac{dv}{dt} = -u\frac{dM}{dt} - M\mathrm{g}}}$

Then $dv = -u\frac{dM}{M} - \mathrm{g}\,dt$

Now if we consider that initial (at $t = 0$) mass and magnitude of velocity of the rocket are $M_0$ & $v_0$ then

$\int_{v_0}^{v(t)} dv = -u\int_{M_0}^{M(t)} \frac{dM}{M} - \int_0^t \mathrm{g}\,dt$

Or, $v(t) - v_0 = -u\ \ln\frac{M(t)}{M_0} - \mathrm{g}t$ $\boxed{\text{or, } v(t) = v_0 - u\ \ln\frac{M(t)}{M_0} - \mathrm{g}t}$ ……………………….(ii)

Then final velocity of the rocket is $v_F = v(T) = v_0 - u\ \ln\frac{M(T)}{M_0} - \mathrm{g}T = v_0 - u\ \ln\frac{M_0 - \alpha T}{M_0} - \mathrm{g}T$

Or, $\boxed{v_F = v_0 - u\ \ln\frac{M'}{M_0} - \mathrm{g}\frac{(M_0 - M')}{\alpha}}$ [ As $M_0 - \alpha T = M'$ & $\mathbf{T} = \frac{(M_0 - M')}{\alpha}$ ]

**<u>Instantaneous height for vertical ascent</u>**

From the equation (ii) $v(t) = \frac{dx(t)}{dt} = v_0 - u\ \ln\frac{M(t)}{M_0} - \mathrm{g}t = v_0 - u\ \ln\frac{M_0 - \alpha t}{M_0} - \mathrm{g}t$

Or, $\int_{x_0}^{x(t)} dx = \int_0^t v_0\,dt - u\int_0^t \ln\frac{M_0 - \alpha t}{M_0}\,dt - \int_0^t -\mathrm{g}t\,\mathrm{d}t$

Or, $x(t) = x_0 + v_0 t + u\left[\left(\frac{M_0 - \alpha t}{\alpha}\ \ln\frac{M_0 - \alpha t}{M_0} + t\right)\right] - \frac{1}{2}\mathrm{g}\,t^2$

**<u>The maximum height reached when the fuel is completely burned out</u>**

$$x_M = x(T) = x_0 + v_0 T + u\left[\left(\frac{M_0 - \alpha T}{\alpha}\ \ln\frac{M_0 - \alpha T}{M_0} + T\right)\right] - \frac{1}{2}\mathrm{g}\,T^2$$

$$= x_0 + v_0 T + u\left[\frac{M'}{\alpha}\ \ln\frac{M'}{M_0} + \frac{(M_0 - M')}{\alpha}\right] - \frac{1}{2}\mathrm{g}\left(\frac{(M_0 - M')}{\alpha}\right)^2$$