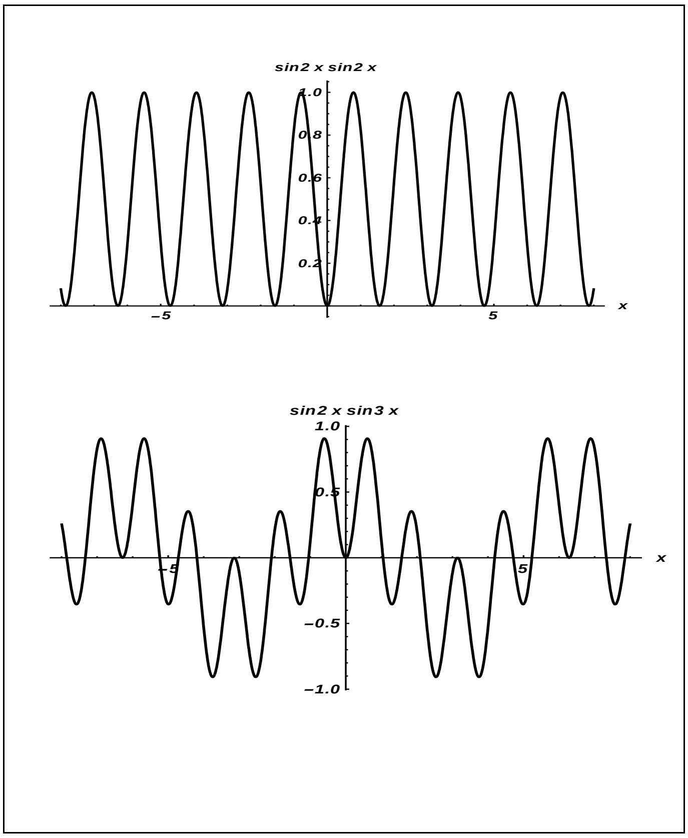
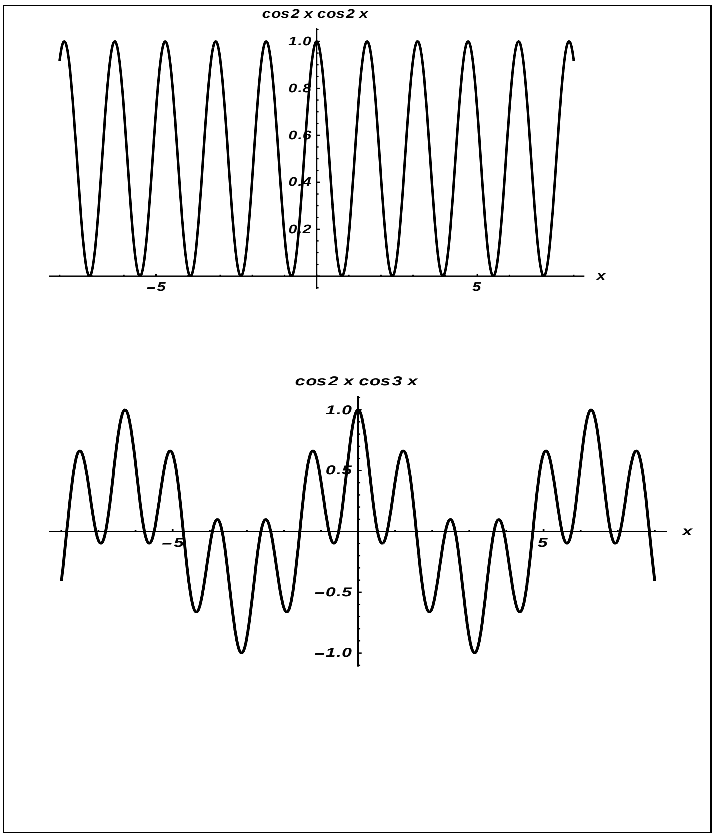
Notes on FOURIER SERIES by Suman Sadhukhan.

1. $\int_{-\pi}^{\pi} \sin mx \sin nx \; dx = \begin{cases} 0, & when\ n = m = 0 \\ \pi, & when\ n = m > 0 \\ 0, & when\ n \neq m \end{cases}$ $\quad \begin{cases} \mathbf{0}, & \boldsymbol{when\ m = 0} \\ \boldsymbol{\pi\delta_{mn}}, & \boldsymbol{when\ m \neq 0} \end{cases}$

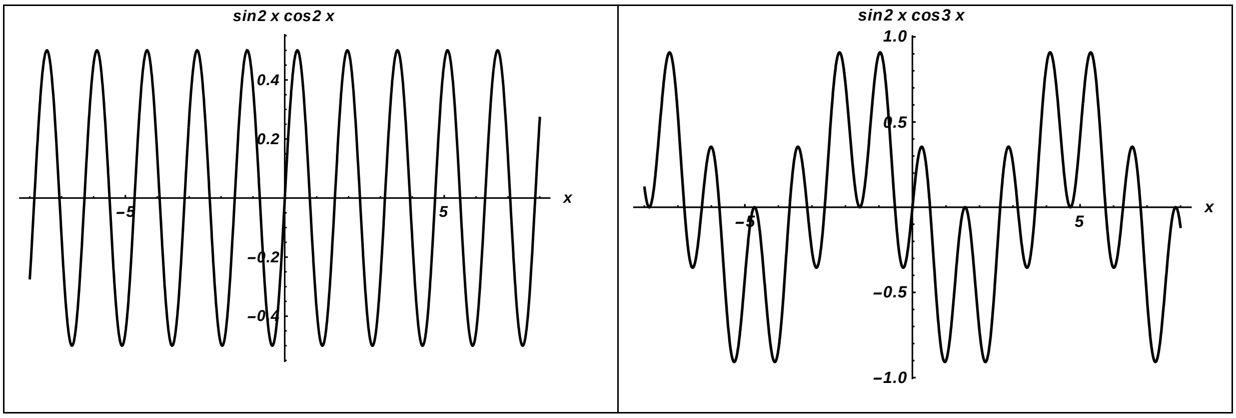
2. $\int_{-\pi}^{\pi} \cos mx \cos nx \; dx = \begin{cases} 2\pi, & when\ n = m = 0 \\ \pi, & when\ n = m > 0 \\ 0, & when\ n \neq m \end{cases}$ $\quad \begin{cases} \boldsymbol{2\pi}, & \boldsymbol{when\ m = 0} \\ \boldsymbol{\pi\delta_{mn}}, & \boldsymbol{when\ m \neq 0} \end{cases}$





3. $\int_{-\pi}^{\pi} \cos mx \sin nx \; dx = \begin{cases} 0, & when\ n = m = 0 \\ 0, & when\ n = m > 0 \\ 0, & when\ n \neq m \end{cases}$

4. $\int_{-\pi}^{\pi} \sin mx \cos nx \; dx = \begin{cases} 0, & when\ n = m = 0 \\ 0, & when\ n = m > 0 \\ 0, & when\ n \neq m \end{cases}$



5. $\int_0^{\pi} \sin mx \sin nx \; dx = \begin{cases} 0\,, & when\, n = m = 0 \\ \pi/2, & when\, n = m > 0 \\ 0, & when\, n \neq m \end{cases}$ $\quad \begin{cases} \mathbf{0}, & \boldsymbol{when\, m = 0} \\ \frac{\pi}{2}\boldsymbol{\delta_{mn}}, & \boldsymbol{when\, m \neq 0} \end{cases}$

6. $\int_0^{\pi} \cos mx \cos nx \; dx = \begin{cases} \pi\,, & when\, n = m = 0 \\ \pi/2, & when\, n = m > 0 \\ 0, & when\, n \neq m \end{cases}$ $\quad \begin{cases} \boldsymbol{\pi}, & \boldsymbol{when\, m = 0} \\ \frac{\pi}{2}\boldsymbol{\delta_{mn}}, & \boldsymbol{when\, m \neq 0} \end{cases}$

7. $\int_0^{\pi} \cos mx \sin nx \; dx = \begin{cases} 0\,, & when\, n = m = 0 \\ 0, & when\, n = m > 0 \\ 0, & when\, n \neq m \end{cases}$

8. $\int_0^{\pi} \sin mx \cos nx \; dx = \begin{cases} 0\,, & when\, n = m = 0 \\ 0, & when\, n = m > 0 \\ 0, & when\, n \neq m \end{cases}$

9. $\int_{-\pi}^{\pi} \sin nx \; dx = 0\,, for\ all\ value\ of\ n$

10. $\int_{-\pi}^{\pi} \cos nx \; dx = \begin{cases} 0\,, & for\, n > 0 \\ 2\pi, & for\, n = 0 \end{cases}$

11. $\boldsymbol{\int_0^{\pi} \sin nx \; dx = \frac{1}{n}\{-(-1)^n + 1\}\ , for\ all\ value\ of\ n}$

$= \begin{cases} \mathbf{0}, \boldsymbol{for\ even\ n} \\ \frac{\mathbf{2}}{\boldsymbol{n}}, \boldsymbol{for\ odd\ n} \end{cases}$

12. $\int_0^{\pi} \cos nx \; dx = \begin{cases} 0\,, & for\, n > 0 \\ \pi, & for\, n = 0 \end{cases}$

13. $\boldsymbol{\int_0^a f(x)g(x)dx = [f(x)g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) - f'''(x)g_4(x) + \cdots \ldots \ldots]_0^a}$

14. Consider a function $f(x)$ defined for all values of x assuming that there exists some positive number T for which

$f(x+T) = f(x)$. The function is called a periodic function and T is called a period.

15. If $f(x)$ is periodic, well defined, bounded, has a finite number of extremum and finite number of discontinuity, then Fourier series for $f(x)$ converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ at every value of x
If $x = x_0$ is a point of discontinuity, then $f(x_0) = \frac{1}{2}[f(x_0+0) + f(x_0-0)]$
where $f(x_0+0) and\ f(x_0-0)$approaches from the right and left respectively.

Dirichlet's Condition

Consider an arbitrary function $f(x)$ which satisfy the following three conditions in the interval(a, b)

1. it is single valued uniformly bounded that is $|f(x)| \leq M$ where $a < x < b$ $and\ M\ its\ constant.$
2. It has finite number of point of discontinuity and all of them are first kind.
3. It is finite number of point of strict extremum.

Statement of Fourier Theorem:

All the periodic function $f(x)$ satisfy the Dirichlet's condition can be expressed by a trigonometric series as

$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ [Wherea_0, a_n, b_n are known as Fourier co-efficients.]

Evolution of coefficient

we have $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ [Wherea_0, a_n, b_n are known as Fourier co-efficients.]

Where $f(x + 2\pi) = f(x)$ for values of x outside of $-\pi\ to\ \pi$

Multiplying both site by dx and integrate over the entire interval $(-\pi\ to\ \pi)$ we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(a_n \cos nx)\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(b_n \sin nx)\,dx$$

$$= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}(a_n) \int_{-\pi}^{\pi} \cos nx\,dx + \sum_{n=1}^{\infty}(b_n) \int_{-\pi}^{\pi} \sin nx\,dx$$

$$= a_0\, 2\pi + 0 + 0$$

Then $\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$

Multiplying both site by $\cos mx\,dx$ and integrate over the entire interval $(-\pi\ to\ \pi)$ we get

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi} a_0 \cos mx\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(a_n \cos nx)\cos mx\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(b_n \sin nx)\cos mx\,dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty}(a_n) \int_{-\pi}^{\pi} \cos mx \cos nx\,dx + \sum_{n=1}^{\infty}(b_n) \int_{-\pi}^{\pi} \cos mx \sin nx\,dx$$

$$= a_0\, 0 + \sum_{n=1}^{\infty}(a_n)\, \boldsymbol{\pi\delta_{mn}} + \sum_{n=1}^{\infty} b_n\ 0 = a_m\, \boldsymbol{\pi}$$

Then $\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$

Multiplying both site by $\sin mx\,dx$ and integrate over the entire interval $(-\pi\ to\ \pi)$ we get

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi} a_0\ \sin mx\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(a_n \cos nx)\sin mx\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(b_n \sin nx)\sin mx\,dx$$

$$= a_0 \int_{-\pi}^{\pi} \sin mx\,dx + \sum_{n=1}^{\infty}(a_n) \int_{-\pi}^{\pi} \sin mx \cos nx\,dx + \sum_{n=1}^{\infty}(b_n) \int_{-\pi}^{\pi} \sin mx\ \sin nx\,dx$$

$$= a_0\, 0 + \sum_{n=1}^{\infty}(a_n)\, \mathbf{0} + \sum_{n=1}^{\infty} b_n\ \boldsymbol{\pi\delta_{mn}} = b_m\, \boldsymbol{\pi}$$

Then $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$

Functions having arbitrary period.

1. Let a function $f(t)$ has a time period T. If we assume the relation between new variable t and x as $t=\frac{T}{2\pi}x$

where the variable x has a time period 2π we have

x	$-\pi$	π
t	-T/2	T/2

Then Fourier Series of the function $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ we have

$$\boxed{f(x)=f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}\right)}$$

Where $f(t+T)=f(t)$ for values of x outside of $-\frac{T}{2}\ to\ \frac{T}{2}$

Now as $t=\frac{T}{2\pi}x$ then $dt=\frac{T}{2\pi}dx$ or, $dx=\frac{2\pi}{T}dt$

Then $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\int_{-T/2}^{T/2}f(t)\,\frac{2\pi}{T}dt=\frac{1}{T}\int_{-T/2}^{T/2}f(t)\,dt$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-T/2}^{T/2}f(t)\cos\frac{2\pi nt}{T}\,\frac{2\pi}{T}dt=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos\frac{2\pi nt}{T}\,dt$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-T/2}^{T/2}f(t)\sin\frac{2\pi nt}{T}\,\frac{2\pi}{T}dt=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin\frac{2\pi nt}{T}\,dt$$

$\boxed{a_0=\frac{1}{T}\int_{-T/2}^{T/2}f(t)\,dt,}$ $\boxed{a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos\frac{2\pi nt}{T}\,dt,}$ $\boxed{b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin\frac{2\pi nt}{T}\,dt}$

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2. Let a function $f(t)$ has a time period 2L. If we assume the relation between new variable t and x as $t=\frac{2L}{2\pi}x$

where the variable x has a time period 2π we have

x	$-\pi$	π
t	$-L$	L

Then Fourier Series of the function $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ we have

$$\boxed{f(x)=f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi nt}{L}+b_n\sin\frac{\pi nt}{L}\right)}$$

Where $f(t+2L)=f(t)$ for values of x outside of $-L\ to\ L$

Now as $t=\frac{2L}{2\pi}x$ then $dt=\frac{L}{\pi}dx$ or, $dx=\frac{\pi}{L}dt$

Then $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\int_{-L}^{L}f(t)\,\frac{\pi}{L}dt=\frac{1}{2L}\int_{-L}^{L}f(t)\,dt$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-L}^{L}f(t)\cos\frac{\pi nt}{L}\,\frac{\pi}{L}dt=\frac{1}{L}\int_{-L}^{L}f(t)\cos\frac{\pi nt}{L}\,dt$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-L}^{L}f(t)\sin\frac{\pi nt}{L}\,\frac{\pi}{L}dt=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{\pi nt}{L}\,dt$$

$\boxed{a_0=\frac{1}{2L}\int_{-L}^{L}f(t)\,dt,}$ $\boxed{a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\frac{\pi nt}{L}\,dt,}$ $\boxed{b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{\pi nt}{L}\,dt}$

Even and odd function

The function $f(x)$ is said to be even if $f(-x) = f(x)$and odd if $f(-x) = -f(x)$

If the function is even

then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\ dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\ dx$

Replacing $x = -x$ in the first integral we get

$$a_0 = \frac{1}{2\pi}\int_{\pi}^{0} f(-x)\ (-dx) + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\ dx = -\frac{1}{2\pi}\int_{\pi}^{0} f(x)\ (dx) + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\ dx = 2 \times \frac{1}{2\pi}\int_{0}^{\pi} f(x)\ dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)\ dx$$

Again, then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx$

Replacing $x = -x$ in the first integral we get

$$a_n = \frac{1}{\pi}\int_{\pi}^{0} f(-x)\ \cos(-nx)(-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx = -\frac{1}{\pi}\int_{\pi}^{0} f(x) \cos nx\ (dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx$$

$$= 2 \times \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx$$

Again, then $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx$

Replacing $x = -x$ in the first integral we get

$$b_n = \frac{1}{\pi}\int_{\pi}^{0} f(-x)\ \sin(-nx)(-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx = \frac{1}{\pi}\int_{\pi}^{0} f(x) \sin nx\ (dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx = 0$$

If the function is odd

then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\ dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\ dx$

Replacing $x = -x$ in the first integral we get

$$a_0 = \frac{1}{2\pi}\int_{\pi}^{0} f(-x)\ (-dx) + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\ dx = \frac{1}{2\pi}\int_{\pi}^{0} f(x)\ (dx) + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\ dx = 0$$

Again, then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx$

Replacing $x = -x$ in the first integral we get

$$a_n = \frac{1}{\pi}\int_{\pi}^{0} f(-x)\ \cos(-nx)(-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx = \frac{1}{\pi}\int_{\pi}^{0} f(x) \cos nx\ (dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx = 0$$

Again, then $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx$

Replacing $x = -x$ in the first integral we get

$$b_n = \frac{1}{\pi}\int_{\pi}^{0} f(-x)\ \sin(-nx)(-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx = -\frac{1}{\pi}\int_{\pi}^{0} f(x) \sin nx\ (dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx$$

$$= 2 \times \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx\ dx$$

For even function	For odd function
$a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)\ dx$, $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx$ $b_n = 0$	$a_0 = 0,$ $a_n = 0$ $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx$

Half range expansion

If $f(x)$ has a time period 2π then for even function $f(x)$ we have $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx)$

If $f(t)$ has a time period T then for even function $f(t)$ we have $f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nt}{T}\right)$

If $f(t)$ has a time period $2L$ then for even function $f(t)$ we have $f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{\pi nt}{L}\right)$

Then $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ or, $a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$ or, $a_0 = \frac{1}{L}\int_0^{L} f(t)\,dt$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$ or, $a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2\pi nt}{T}\,dt$ or, $a_n = \frac{2}{L}\int_0^{L} f(t)\cos\frac{\pi nt}{L}\,dt$

Similarly If $f(x)$ has a time period 2π then for odd function $f(x)$ we have $f(x) = \sum_{n=1}^{\infty}(b_n \sin nx)$

If $f(t)$ has a time period T then for odd function $f(t)$ we have $f(t) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{2\pi nt}{T}\right)$

If $f(t)$ has a time period $2L$ then for odd function $f(t)$ we have $f(t) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{\pi nt}{L}\right)$

Then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ or, $b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2\pi nt}{T}\,dt$ or, $b_n = \frac{2}{L}\int_0^{L} f(t)\sin\frac{\pi nt}{L}\,dt$

PERSEVAL'S IDENTITY FOR FOURIER SERIES.

With period 2π the Fourier expansion of a function $f(x)$ is given by $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Multiplying both side by $f(x)$ We get $f(x)f(x) = a_0 f(x) + \sum_{n=1}^{\infty}(a_n f(x)\cos nx + b_n f(x)\sin nx)$

Integrating both side w.r.t x within the limit $-\pi$ *to* π we get

$$\int_{-\pi}^{\pi} f(x)f(x)\,dx = \int_{-\pi}^{\pi} a_0 f(x)dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_n f(x)\cos nx)\,dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(b_n f(x)\sin nx)\,dx$$

$$= a_0\int_{-\pi}^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}(a_n)\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \sum_{n=1}^{\infty}(b_n)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= a_0 a_0\,2\pi + \sum_{n=1}^{\infty}(a_n)(a_n)\pi + \sum_{n=1}^{\infty}(b_n)(b_n)\pi$$

Then $\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = {a_0}^2 + \frac{1}{2}\left({a_n}^2 + {b_n}^2\right)}$

Similarly for function $f(t)$ with period T and period 2L we have $\boxed{\frac{1}{T}\int_{-T/2}^{T/2}|f(t)|^2\,dt = {a_0}^2 + \frac{1}{2}\left({a_n}^2 + {b_n}^2\right)}$

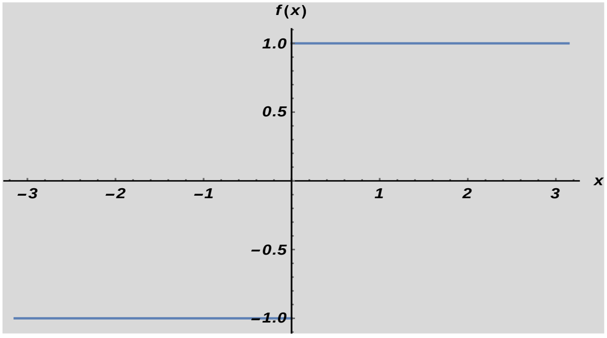
& $\boxed{\frac{1}{2L}\int_{-L}^{L}|f(t)|^2\,dt = {a_0}^2 + \frac{1}{2}\left({a_n}^2 + {b_n}^2\right)}$

What is the advantage of Fourier expansion over the Taylor's expansion?

The Fourier series can be used to represent discontinuous function where all orders of derivatives need not exist. Whereas Taylor's expansion is possible in case all orders of derivative do exists.

Q.1 Find the Fourier series expansion of the function (square wave) mathematically defined as

$$\boldsymbol{f(x) = \begin{cases} -K, & for -\pi < x < 0 \\ K, & for \quad 0 < x < \pi \end{cases}}$$ **where $f(x+2\pi) = f(x)$ And show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.**



Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -K\,dx + \frac{1}{2\pi}\int_{0}^{\pi} K\,dx = -\frac{K}{2\pi}(0-(-\pi)) + \frac{K}{2\pi}(\pi - 0) = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -K\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} K\cos nx\,dx$

$$= -\frac{K}{\pi}\int_{-\pi}^{0}\cos nx\,dx + \frac{K}{\pi}\int_{0}^{\pi}\cos nx\,dx = -\frac{K}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{K}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi} = 0$$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -K\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} K\sin nx\,dx$

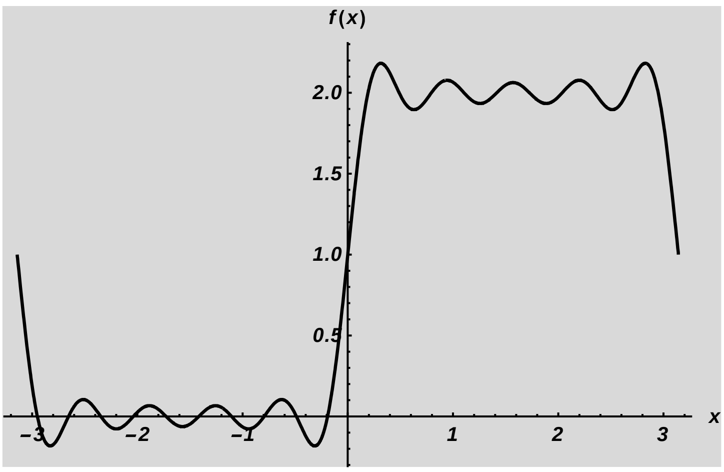
$$= -\frac{K}{\pi}\int_{-\pi}^{0}\sin nx\,dx + \frac{K}{\pi}\int_{0}^{\pi}\sin nx\,dx = -\frac{K}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{K}{\pi}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$= -\frac{K}{n\pi}(-1+\cos n\pi) + \frac{K}{\pi}(-\cos n\pi + 1) = \frac{2k}{n\pi}\{1-(-1)^n\}$$

$$b_n = \begin{cases} \frac{4k}{n\pi}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

$f(x) = 0 + \sum_{n=1}^{\infty} 0 + \frac{2k}{n\pi}\{1-(-1)^n\}\sin nx = \sum_{1,3,5,7\ldots} \frac{4k}{n\pi}\sin nx$

Hence $f(x) = \frac{4k}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots \ldots \ldots \ldots\right)$

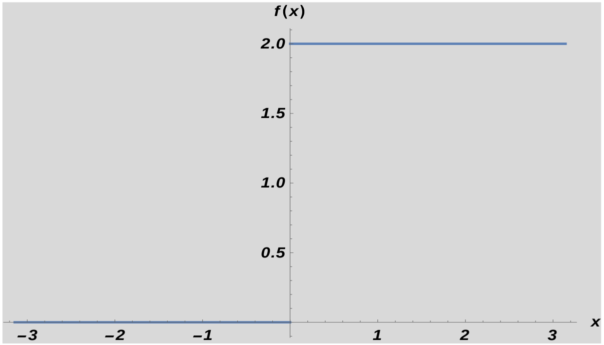


If we put $x = \frac{\pi}{2}$ then $f(\pi/2) = K$ then from above equation $K = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.\right)$

Hence $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. (proved)

Q.2 Find the Fourier series expansion of the function (rectangular pulse wave) mathematically defined as

$f(x) = \begin{cases} 0, & for -\pi < x < 0 \\ 2K, & for \quad 0 < x < \pi \end{cases}$ where $f(x+2\pi) = f(x)$ And show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$



Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2K\,dx = 0 + \frac{2K}{2\pi}(\pi - 0) = K$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} 2K\cos nx\;dx$

$$= 0 \quad + \frac{2K}{\pi}\int_{0}^{\pi}\cos nx\;dx \;=\; \frac{2K}{\pi}\left[\frac{\sin\,nx}{n}\right]_0^{\pi} = 0$$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} 2K\sin nx\;dx$

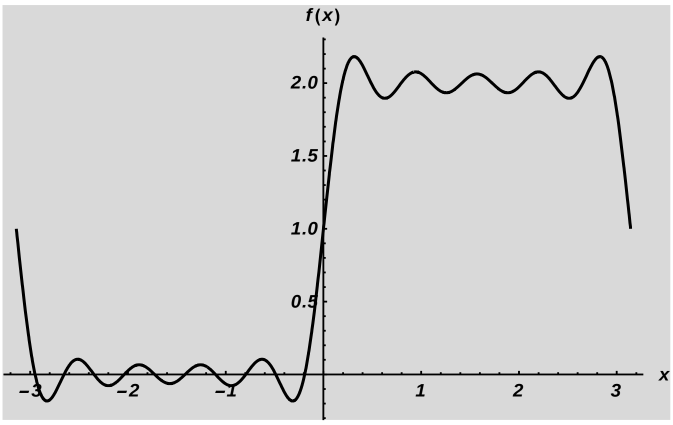
$$= 0 + \frac{2K}{\pi}\int_{0}^{\pi}\sin nx\;dx \;= \frac{2K}{\pi}\left[-\frac{\cos\,nx}{n}\right]_0^{\pi}$$

$$= \frac{2K}{n\,\pi}(-\cos\,n\pi + 1) = \frac{2k}{n\,\pi}\{1 - (-1)^n\}$$

$$b_n = \begin{cases} \frac{4k}{n\,\pi}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

$f(x) = \mathrm{K} + \sum_{n=1}^{\infty} 0 + \frac{2k}{n\,\pi}\{1-(-1)^n\}\sin nx = K + \sum_{1,3,5,7\ldots.}\frac{4k}{n\,\pi}\sin nx$

Hence $f(x) = K + \frac{4k}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\ldots\ldots\ldots\right)$



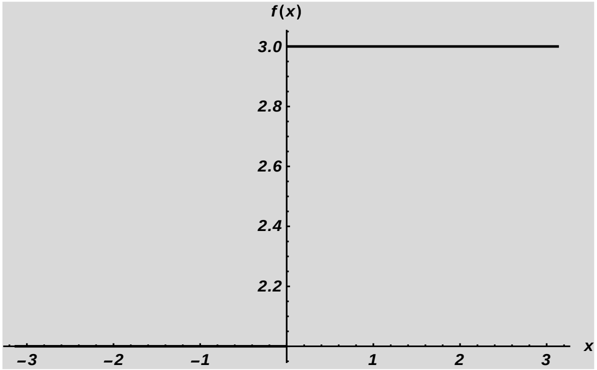
If we put $x = \frac{\pi}{2}$ then $f(\pi/2) = 2K$ then from above equation $2K = K + \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.\right)$

$$K = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.\right)$$

Hence $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$ (proved)

Q.3 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = \begin{cases} a, & for -\pi < x < 0 \\ b, & for \quad 0 < x < \pi \end{cases} \qquad \text{where } f(x+2\pi) = f(x)$$



Taking $a = 2$ and $b = 3$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} a\,dx + \frac{1}{2\pi}\int_{0}^{\pi} b\,dx = -\frac{a}{2\pi}(0-(-\pi) + \frac{b}{2\pi}(\pi - 0) = \frac{1}{2}(a+b)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} a\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} b\cos nx\;dx$$

$$= \frac{a}{\pi}\int_{-\pi}^{0}\cos nx\;dx + \frac{b}{\pi}\int_{0}^{\pi}\cos nx\;dx = \frac{a}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{b}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi} = 0$$

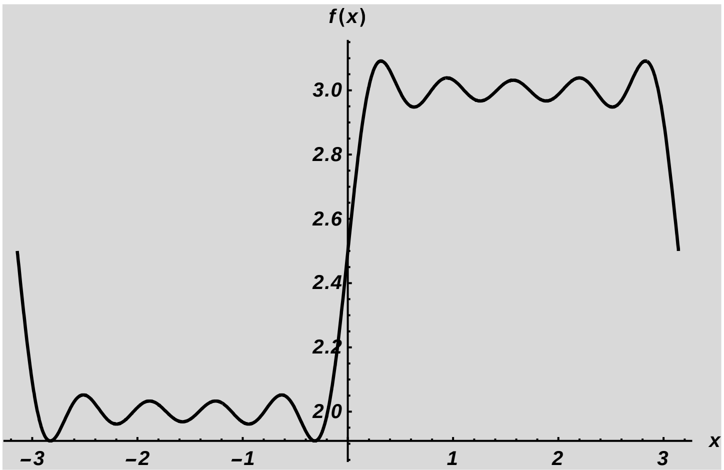
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} a\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} b\sin nx\;dx$$

$$= \frac{a}{\pi}\int_{-\pi}^{0}\sin nx\;dx + \frac{b}{\pi}\int_{0}^{\pi}\sin nx\;dx = \frac{a}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{b}{\pi}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{a}{n\,\pi}(-1+\cos n\pi) + \frac{b}{\pi}(-\cos n\pi + 1) = \frac{(b-a)}{n\,\pi}\{1-(-1)^n\}$$

$$b_n = \begin{cases} \dfrac{2(b-a)}{n\,\pi}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

$$f(x) = \frac{1}{2}(a+b) + \sum_{n=1}^{\infty} 0 + \frac{2k}{n\,\pi}\{1-(-1)^n\}\sin nx = \frac{1}{2}(a+b) + \sum_{1,3,5,7....}\frac{2(b-a)}{n\,\pi}\sin nx$$



Taking $a = 2$ and $b = 3$

Q.4 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = A, \quad for \quad 0 < x < \pi$$ where $f(x + 2\pi) = f(x)$ And show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Then $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} A\, dx = A$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} A\cos nx\ dx = \frac{2A}{\pi}\int_0^{\pi}\cos nx\ dx = \frac{2A}{\pi}\left[\frac{\sin\ nx}{n}\right]_0^{\pi} = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx = \frac{2}{\pi}\int_0^{\pi} A\sin nx\ dx = \frac{2A}{\pi}\int_0^{\pi}\sin nx\ dx = \frac{2A}{\pi}\left[-\frac{\cos\ nx}{n}\right]_0^{\pi}$$

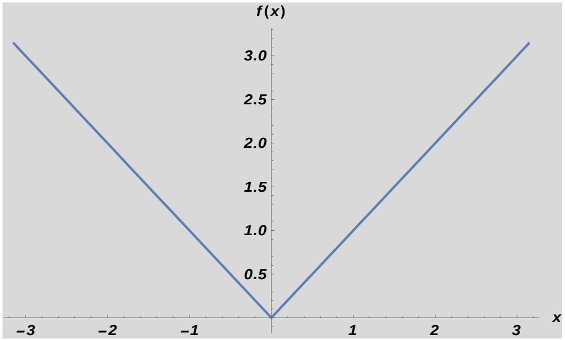
$$= \frac{2A}{n\,\pi}(-\cos\ n\pi + 1) = \frac{2A}{n\,\pi}\{1 - (-1)^n\}$$

$$b_n = \begin{cases} \dfrac{4A}{n\,\pi}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

Q.5 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = \begin{cases} -x, & for -\pi < x < 0 \\ x, & for \quad 0 < x < \pi \end{cases} \quad \textbf{that means} \quad \boldsymbol{f(x) = |x|, \quad for \quad -\pi < x < \pi}$$

where $f(x+2\pi) = f(x)$ And show that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \dots\dots\dots$



Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -x\,dx + \frac{1}{2\pi}\int_0^{\pi} x\,dx = -\frac{1}{2\pi}\left(\frac{\pi^2}{2} + \frac{\pi^2}{2}\right) = \boldsymbol{\frac{\pi}{2}}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx \quad \dots\dots\dots\dots\text{(i)}$$

$$= -\frac{1}{\pi}\left[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\left(0 - \frac{-1}{n^2}\right) - \left(0 - \frac{-(-1)^n}{n^2}\right)\right] + \frac{1}{\pi}\left[\left(0 - \frac{-(-1)^n}{n^2}\right) - \left(0 - \frac{-1}{n^2}\right)\right] = \frac{2}{\pi n^2}((-1)^n - 1)$$

Another method

Replacing $x = -x$ in the first integral in equatiobn (i)we get

$$a_n = -\frac{1}{\pi}\int_{\boldsymbol{\pi}}^{0} -\boldsymbol{x}\cos(-\boldsymbol{nx})\;(-\boldsymbol{dx}) + \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = 2\times\frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(0 - \frac{-(-1)^n}{n^2}\right) - \left(0 - \frac{-1}{n^2}\right)\right] = \frac{2}{\pi n^2}((-1)^n - 1)$$

$$a_n = \begin{cases} -\frac{4}{\pi n^2}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx \quad \dots\dots\dots\dots\dots\text{(ii)}$$

$$= -\frac{1}{\pi}\left[x\frac{-\cos nx}{n} - \frac{-\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\frac{-\cos nx}{n} - \frac{-\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[(0-0) - \left(\frac{(-\pi)(-(-1)^n)}{n} - 0\right)\right] + \frac{1}{\pi}\left[\left(\frac{(\pi)(-(-1)^n)}{n} - 0\right) - (0-0)\right] = 0$$

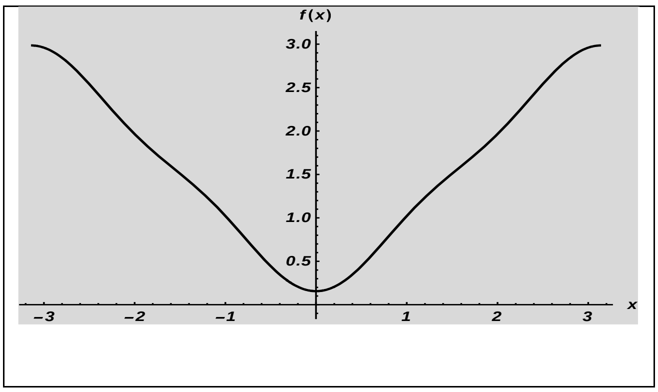
Another method

Replacing $x = -x$ in the first integral in equatiobn (ii)we get

$$b_n = -\frac{1}{\pi}\int_{\pi}^{0} -x\sin(-nx)\;(-dx) + \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = 0$$

Then $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\left\{\frac{2}{\pi n^2}((-1)^n - 1)\right\}\cos nx + 0\sin nx\right) = \frac{\pi}{2} + \sum_{1,3,5,7\dots} -\frac{4}{\pi n^2}\cos nx$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2}\dots\dots\right)$$



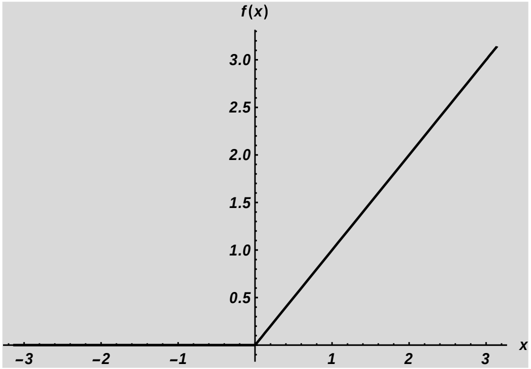
Now putting $x = 0$ we have f(x)=0

Then from above equation 0 $= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\dots\dots\right)$

Then $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \dots\dots\dots.(proved)$

Q.6 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = \begin{cases} 0, & for -\pi \le x \le 0 \\ x, & for \quad 0 \le x \le \pi \end{cases} \quad \text{where } f(x+2\pi) = f(x) \quad \text{And show that } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$



Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx = \frac{1}{2\pi}(\frac{\pi^2}{2}) = \boldsymbol{\frac{\pi}{4}}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\;dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\;dx = \frac{1}{\pi}\left[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\right]_0^{\pi} = \frac{1}{\pi}\left[\left(0 - \frac{-(-1)^n}{n^2}\right) - \left(0 - \frac{-1}{n^2}\right)\right] = \frac{1}{\pi n^2}((-1)^n - 1)$$

$$a_n = \begin{cases} -\frac{2}{\pi n^2}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\;\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\;dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\;dx = \frac{1}{\pi}\left[x\frac{-\cos nx}{n} - \frac{-\sin nx}{n^2}\right]_0^{\pi} = \frac{1}{\pi}\left[\left(\frac{(\pi)(-(-1)^n)}{n} - 0\right) - (0-0)\right] = \frac{-(-1)^n}{n}$$

$$b_n = \begin{cases} \frac{1}{n}, & for\ odd\ n \\ -\frac{1}{n} & for\ even\ n \end{cases}$$

Then $f(x) = \frac{\boldsymbol{\pi}}{\boldsymbol{4}} + \sum_{n=1}^{\infty}\left(\left\{\frac{1}{\pi n^2}((-1)^n - 1)\right\}\cos nx + \frac{-(-1)^n}{n}\sin nx\right)$

$$= \frac{\boldsymbol{\pi}}{\boldsymbol{4}} + \sum\nolimits_{\mathbf{1,3,5,7\ldots}} -\frac{2}{\pi n^2}\cos nx + \sum\nolimits_{\mathbf{1,3,5\ldots}}\frac{\sin nx}{n} - \sum\nolimits_{\mathbf{2,4,6\ldots}}\frac{\sin nx}{n}$$

$$f(x) = \frac{\boldsymbol{\pi}}{\boldsymbol{4}} - \frac{\mathbf{2}}{\boldsymbol{\pi}}\left(\frac{\mathbf{cos}\,x}{\mathbf{1^2}} + \frac{\mathbf{cos\,3}x}{\mathbf{3^2}} + \frac{\mathbf{cos\,5}x}{\mathbf{5^2}} + \frac{\mathbf{cos\,7}x}{\mathbf{7^2}}\ldots..\right) + \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\ldots\ldots\ldots\ldots\right) - \left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6}\ldots\ldots\ldots\ldots\right)$$

Now putting $\boldsymbol{x = \pi}$ we have $\boldsymbol{f(\pi)} = \frac{1}{2}[f(\boldsymbol{\pi} + 0) + f(\boldsymbol{\pi} - 0)] = \frac{1}{2}(0+\pi) = \frac{\pi}{2}$

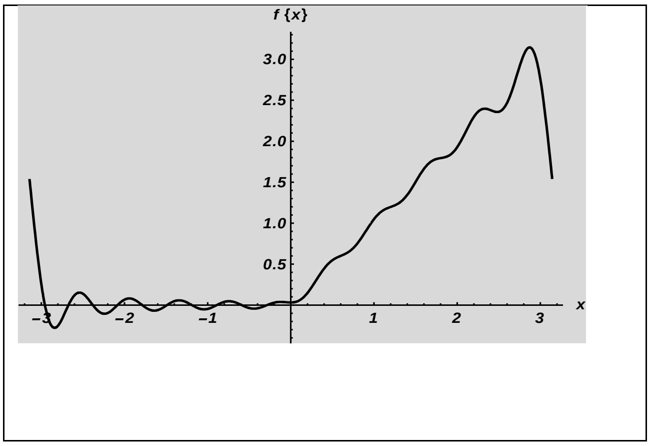
Then from above equation $\frac{\pi}{2} = \frac{\boldsymbol{\pi}}{\boldsymbol{4}} + \frac{\mathbf{2}}{\boldsymbol{\pi}}\left(\frac{\mathbf{1}}{\mathbf{1^2}} + \frac{\mathbf{1}}{\mathbf{3^2}} + \frac{\mathbf{1}}{\mathbf{5^2}} + \frac{\mathbf{1}}{\mathbf{7^2}}\ldots..\right) + \mathbf{0} + \mathbf{0}$

Then $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \ldots\ldots\ldots.(proved)$

Or, putting $\boldsymbol{x = 0}$ we have $\boldsymbol{f(0)} = 0$

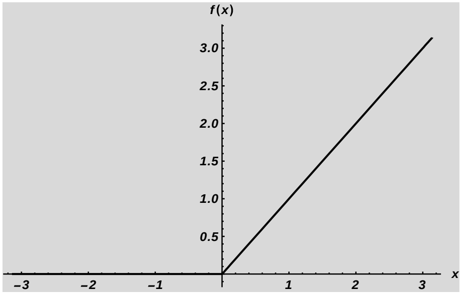
Then from above equation $0 = \frac{\boldsymbol{\pi}}{\boldsymbol{4}} - \frac{\mathbf{2}}{\boldsymbol{\pi}}\left(\frac{\mathbf{1}}{\mathbf{1^2}} + \frac{\mathbf{1}}{\mathbf{3^2}} + \frac{\mathbf{1}}{\mathbf{5^2}} + \frac{\mathbf{1}}{\mathbf{7^2}}\ldots..\right) + \mathbf{0} + \mathbf{0}$

Then $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \ldots\ldots\ldots.(proved)$



Q.7 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = \begin{cases} 0, & for -l \le x \le 0 \\ x, & for \quad 0 \le x \le l \end{cases} \quad \text{where } f(x+2l) = f(x) \quad \text{And show that } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$



Then $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2l}\int_{-l}^{0} f(x)\,dx + \frac{1}{2l}\int_{0}^{l} f(x)\,dx = \frac{1}{2l}\int_{-l}^{0} 0\,dx + \frac{1}{2l}\int_{0}^{l} x\,dx = \frac{1}{2l}(\frac{l^2}{2}) = \boldsymbol{\frac{l}{4}}$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{0} f(x)\cos\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{0} 0\cos\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l} x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{0}^{l} x\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[x\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l = \frac{1}{l}\left[\left(0 - \frac{-(-1)^n}{\left(\frac{n\pi}{l}\right)^2}\right) - \left(0 - \frac{-1}{\left(\frac{n\pi}{l}\right)^2}\right)\right] = \frac{l}{\pi^2 n^2}((-1)^n - 1)$$

$$a_n = \begin{cases} -\frac{2l}{\pi^2 n^2}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{0} f(x)\sin\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{0} 0\ \sin\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l} x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{0}^{l} x\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[x\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l = \frac{1}{l}\left[\left(\frac{(l)(-(-1)^n)}{\frac{n\pi}{l}} - 0\right) - (0-0)\right] = \frac{-l(-1)^n}{\pi n} = \frac{l(-1)^{n+1}}{\pi n}$$

$$b_n = \begin{cases} \frac{l}{n\pi}, & for\ odd\ n \\ -\frac{l}{n\pi} & for\ even\ n \end{cases}$$

Then $f(x) = \frac{\boldsymbol{l}}{\boldsymbol{4}} + \sum_{n=1}^{\infty}\left(\left\{\frac{l}{\pi^2 n^2}((-1)^n - 1)\right\}\cos\frac{n\pi x}{l} + \frac{l(-1)^{n+1}}{\pi n}\sin\frac{n\pi x}{l}\right)$

$$= \frac{\boldsymbol{l}}{\boldsymbol{4}} + \sum_{\mathbf{1,3,5,7...}} -\frac{2l}{\pi^2 n^2}\cos\frac{n\pi x}{l} + \frac{l}{\pi}\sum_{1,3,5...}\frac{\sin nx}{n} - \frac{l}{\pi}\sum_{2,4,6...}\frac{\sin\frac{n\pi x}{l}}{n}$$

$$f(x) = \frac{\boldsymbol{l}}{\boldsymbol{4}} - \frac{\boldsymbol{2l}}{\boldsymbol{\pi^2}}\left(\frac{\cos\frac{x\pi}{l}}{1^2} + \frac{\cos 3\frac{x\pi}{l}}{3^2} + \frac{\cos 5\frac{x\pi}{l}}{5^2} + \frac{\cos 7\frac{x\pi}{l}}{7^2}\dots\right) + \frac{l}{\pi}\left(\frac{\sin\frac{x\pi}{l}}{1} + \frac{\sin 3\frac{x\pi}{l}}{3} + \frac{\sin 5\frac{x\pi}{l}}{5}\dots\dots\dots\right) - \frac{l}{\pi}\left(\frac{\sin 2\frac{x\pi}{l}}{2} + \frac{\sin 4\frac{x\pi}{l}}{4} + \frac{\sin 6\frac{x\pi}{l}}{6}\dots\right)$$

Now putting $\boldsymbol{x = l}$ we have $\boldsymbol{f(l)} = \frac{1}{2}[f(\boldsymbol{l}+0) + f(\boldsymbol{l}-0)] = \frac{1}{2}(0+l) = \frac{l}{2}$

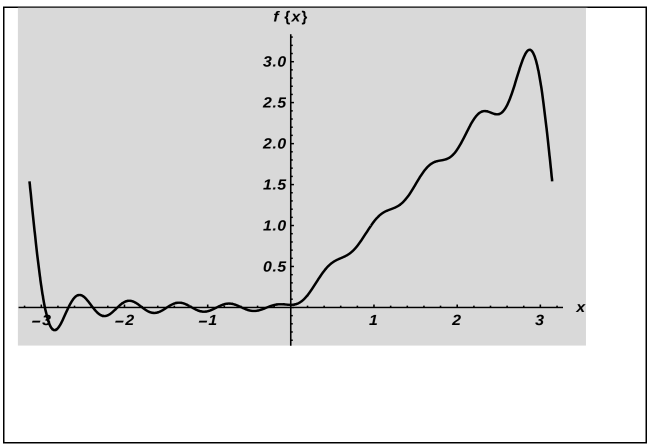
Then from above equation $\frac{l}{2} = \frac{\boldsymbol{l}}{\boldsymbol{4}} + \frac{\boldsymbol{2l}}{\boldsymbol{\pi^2}}\left(\frac{\mathbf{1}}{\mathbf{1^2}} + \frac{\mathbf{1}}{\mathbf{3^2}} + \frac{\mathbf{1}}{\mathbf{5^2}} + \frac{\mathbf{1}}{\mathbf{7^2}}\dots\right) + \mathbf{0} + \mathbf{0}$

Then $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \dots\dots\dots.(proved)$

Or, putting $\boldsymbol{x = 0}$ we have $\boldsymbol{f(0)} =$ 0

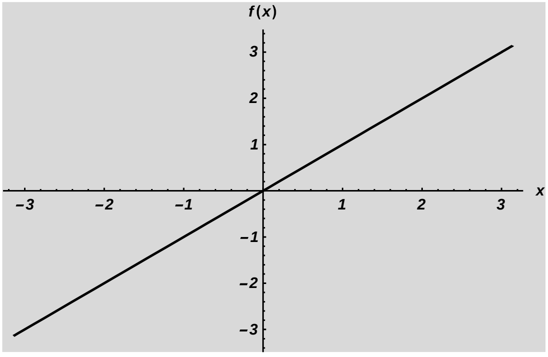
Then from above equation $0 = \frac{\boldsymbol{l}}{\boldsymbol{4}} - \frac{\boldsymbol{2l}}{\boldsymbol{\pi^2}}\left(\frac{\mathbf{1}}{\mathbf{1^2}} + \frac{\mathbf{1}}{\mathbf{3^2}} + \frac{\mathbf{1}}{\mathbf{5^2}} + \frac{\mathbf{1}}{\mathbf{7^2}}\dots\right) + \mathbf{0} + \mathbf{0}$

Then $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \dots\dots\dots.(proved)$



Q.8 Find the Fourier series expansion of the function mathematically defined as

$f(x) = x, \quad for \quad -\pi < x < \pi$ where $f(x+2\pi) = f(x)$



As $f(-x) = -x$ then function is odd. Then $a_0 = 0, a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \; dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx \; dx$$

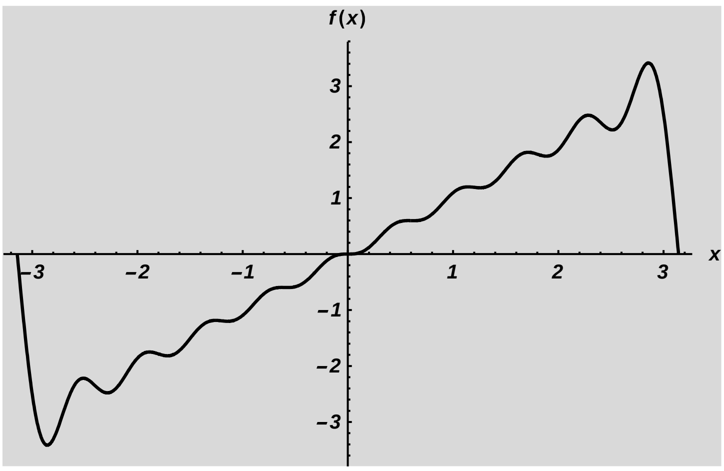
$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx \; dx = \frac{2}{\pi}\left[x\frac{-\cos nx}{n} - \frac{-\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{(\pi)(-(-1)^n)}{n} - 0\right) - (0-0)\right] = \frac{-2(-1)^n}{n}$$

$$b_n = \begin{cases} \frac{2}{n}, & for\ odd\ n \\ -\frac{2}{n} & for\ even\ n \end{cases}$$

Then $f(x) = \mathbf{0} + \sum_{n=1}^{\infty}\left(0.\cos nx + \frac{-2(-1)^n}{n}\sin nx\right)$

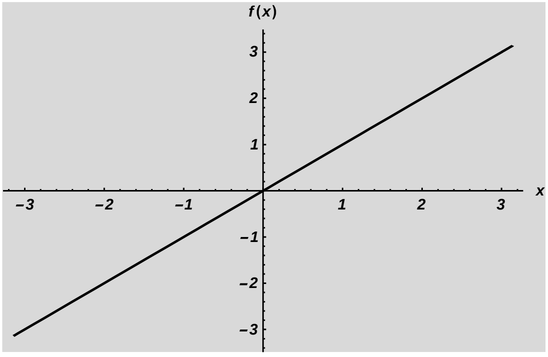
$$= \sum_{1,3,5\ldots}\frac{2\sin nx}{n} - \sum_{2,4,6\ldots}\frac{2\sin nx}{n}$$

$$f(x) = \mathbf{2}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \ldots\ldots\ldots\ldots\right) - \mathbf{2}\left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} \ldots\ldots\ldots\ldots\right)$$



Q.9 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = x, \quad for \quad -l < x < l \qquad \text{where } f(x+2l) = f(x)$$



As $f(-x) = -x$ then function is odd. Then $a_0 = 0, a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

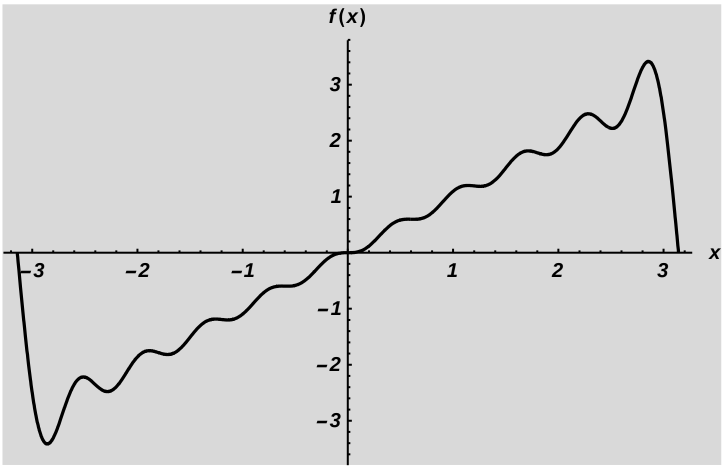
$$= \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[x\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l = \frac{2}{l}\left[\left(\frac{(l)(-(-1)^n)}{\frac{n\pi}{l}} - 0\right) - (0-0)\right] = \frac{-2l(-1)^n}{n\pi} = \frac{2l(-1)^{n+1}}{n\pi}$$

$$b_n = \begin{cases} \frac{2}{n}, & for\ odd\ n \\ -\frac{2}{n} & for\ even\ n \end{cases}$$

Then $f(x) = \mathbf{0} + \sum_{n=1}^{\infty}\left(0.\cos\frac{n\pi x}{l} + \frac{2l(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\right)$

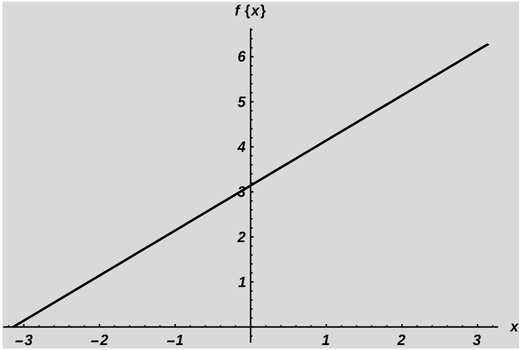
$$= \frac{l}{\pi}\sum_{1,3,5\ldots}\frac{2\sin\frac{n\pi x}{l}}{n} - \frac{l}{\pi}\sum_{2,4,6\ldots}\frac{2\sin\frac{n\pi x}{l}}{n}$$

$$f(x) = \mathbf{2}\frac{l}{\pi}\left(\frac{\sin x\frac{\pi}{l}}{1} + \frac{\sin 3x\frac{\pi}{l}}{3} + \frac{\sin 5x\frac{\pi}{l}}{5} \ldots\ldots\ldots\ldots\right) - \mathbf{2}\frac{l}{\pi}\left(\frac{\sin 2x\frac{\pi}{l}}{2} + \frac{\sin 4x\frac{\pi}{l}}{4} + \frac{\sin 6x\frac{\pi}{l}}{6} \ldots\ldots\ldots\ldots\right)$$



Q.10 Find the Fourier series expansion of the function mathematically defined as

$f(x) = x + \pi, \quad for \quad -\pi < x < \pi$ where $f(x + 2\pi) = f(x)$ And show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$



Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + \pi)dx = \frac{1}{2\pi}\left\{\left(\frac{\pi^2}{2} + \pi^2\right) - \left(\frac{\pi^2}{2} - \pi^2\right)\right\} = \boldsymbol{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\; dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + \pi) \cos nx\; dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\; \cos nx\; dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\; (\pi) \cos nx\; dx$

$= \frac{1}{\pi} \quad \int_{-\pi}^{\pi} x \cos nx\; dx \quad + 0$

$= \frac{1}{\pi}\;\left[x\frac{\sin\, nx}{n} - \frac{-\cos\, nx}{n^2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\;\left[\left(0 - \frac{-(-1)^n}{n^2}\right)\right] - \frac{1}{\pi}\left[\left(0 - \frac{-(-1)^n}{n^2}\right)\right] = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\; dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + \pi) \sin nx\; dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\; \sin nx\; dx + \frac{1}{\pi}\int_{-\pi}^{\pi} \pi \sin nx\; dx$

$= \frac{1}{\pi} \quad \int_{-\pi}^{\pi} x \sin nx\; dx \quad +0 \quad = \frac{1}{\pi}\;\left[x\frac{-\cos\, nx}{n} - \frac{-\sin\, nx}{n^2}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\;\left[\left(\frac{(\pi)(-(-1)^n)}{n} - 0\right) - \left(\frac{(-\pi)(-(-1)^n)}{n} - 0\right)\right] = \frac{-2(-1)^n}{n}$

$$b_n = \begin{cases} \frac{2}{n}, & for\ odd\ n \\ -\frac{2}{n} & for\ even\ n \end{cases}$$

Then $f(x) = \boldsymbol{\pi} + \sum_{n=1}^{\infty}\left(0.\cos nx + \frac{-2(-1)^n}{n}\sin nx\right)$

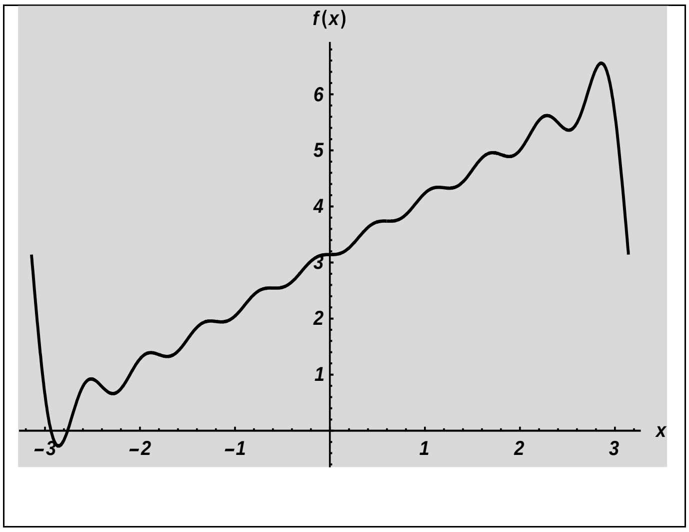
$= \boldsymbol{\pi} + \sum_{1,3,5\ldots}\frac{2\sin nx}{n} - \sum_{2,4,6\ldots}\frac{2\sin nx}{n}$

$f(x) = \boldsymbol{\pi} + \mathbf{2}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} \ldots\ldots\ldots\ldots\right) - \mathbf{2}\left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} \ldots\ldots\ldots\ldots\right)$

Now putting $\boldsymbol{x = \frac{\pi}{2}}$ we have $\boldsymbol{f(x) = \frac{3\pi}{2}}$

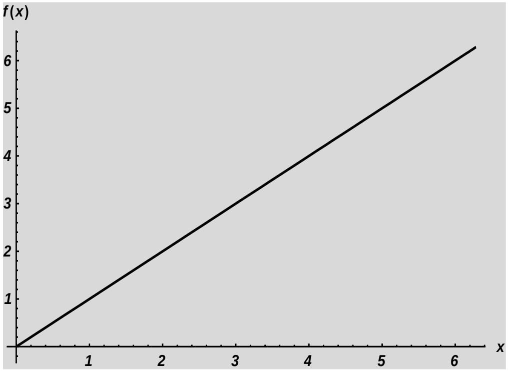
Then from above equation $\frac{3\pi}{2} = \boldsymbol{\pi} + \mathbf{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots\ldots\ldots\ldots\right) + 0$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ (proved)



Q.11 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = x, \quad for \quad 0 < x < 2\pi \text{ where } f(x + 2\pi) = f(x) \text{ And show that } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$



Then $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\, dx = \boldsymbol{\pi}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\ dx = \frac{1}{\pi}\int_0^{2\pi} x\ \cos nx\ dx\ = \frac{1}{\pi}\left[x\frac{\sin\ nx}{n} - \frac{-\cos\ nx}{n^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[\left(0 - \frac{-1}{n^2}\right)\right] - \frac{1}{\pi}\left[\left(0 - \frac{-1}{n^2}\right)\right] = 0$

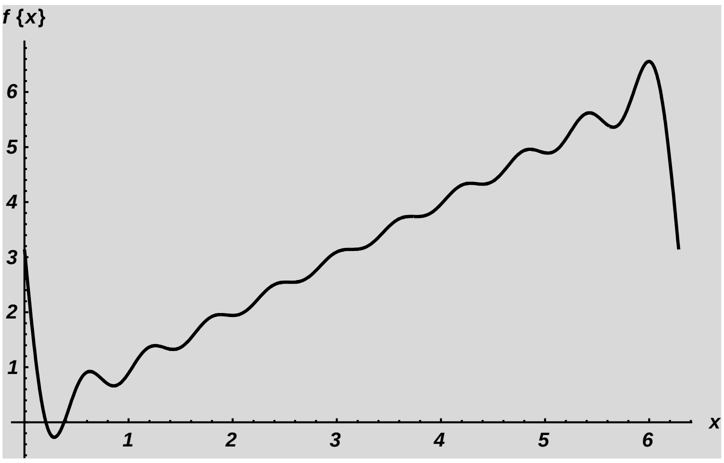
$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\ dx = \frac{1}{\pi}\int_0^{2\pi} x\ \sin nx\ dx = \frac{1}{\pi}\left[x\frac{-\cos\ nx}{n} - \frac{-\sin\ nx}{n^2}\right]_0^{2\pi}$

$$= \frac{1}{\pi}\left[\left(\frac{(2\pi)(-1)}{n} - 0\right) - (0 - 0)\right] = \frac{-2}{n}$$

Then $f(x) = \boldsymbol{\pi} + \sum_{n=1}^{\infty}\left(0.\cos nx + \frac{-2}{n}\sin nx\right)$

$$= \boldsymbol{\pi} - \sum_{1,2,3}\frac{2\sin nx}{n}$$

$$f(x) = \boldsymbol{\pi} - \mathbf{2}\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \frac{\sin 6x}{6} + \frac{\sin 7x}{7} \ldots \ldots \ldots \ldots\right)$$



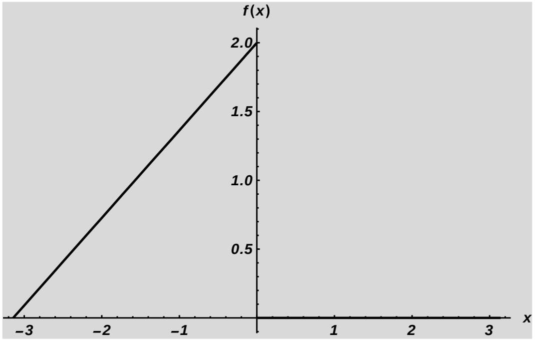
Now putting $x = \frac{\pi}{2}$ we have $f(x) = \frac{\pi}{2}$

Then from above equation $\frac{\pi}{2} = \boldsymbol{\pi} - \mathbf{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \ldots \ldots \ldots\right) + 0$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ (proved)

Q.12 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = \begin{cases} \frac{ax}{\pi} + a, & for - \pi < x < 0 \\ 0, & for \quad 0 < x < \pi \end{cases} \quad \text{where } f(x + 2\pi) = f(x)$$



taking $a = 2$

Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0}(\frac{ax}{\pi} + a)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 0\,dx = \frac{1}{2\pi}\{0 - (-\frac{a}{\pi}\frac{\pi^2}{2} + \frac{a(-\pi)}{2}) = \boldsymbol{\frac{a}{4}}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0}(\frac{ax}{\pi} + a)\;\cos nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} 0\cos nx\;dx$

$= \frac{1}{\pi}\;\frac{a}{\pi}\;\int_{-\pi}^{0} x\cos nx\;dx \quad + \frac{1}{\pi}\;\int_{-\pi}^{0} a\cos nx\;dx$

$= \frac{1}{\pi}\;\frac{a}{\pi}\;\left[x\frac{\sin\,nx}{n} - \frac{-\cos\,nx}{n^2}\right]_{-\pi}^{0} + \;\frac{a}{\pi}\;\;\int_{-\pi}^{0}\cos nx\;dx$

$= \frac{1}{\pi}\frac{a}{\pi}\left[\left(0 - \frac{-1}{n^2}\right) - \left(0 - \frac{-(-1)^n}{n^2}\right)\right] + \;0 = \frac{a}{\pi^2 n^2}(1 - (-1)^n)$ then $a_n = \begin{cases} \frac{2a}{\pi^2 n^2}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\;dx = \frac{1}{\pi}\int_{-\pi}^{0}(\frac{ax}{\pi} + a)\sin nx\;dx + \frac{1}{\pi}\int_{0}^{\pi} 0\sin nx\;dx$

$= \frac{1}{\pi}\;\frac{a}{\pi}\;\;\int_{-\pi}^{0} x\sin nx\;dx \quad + \frac{1}{\pi}\;\int_{-\pi}^{0} a\sin nx\;dx$

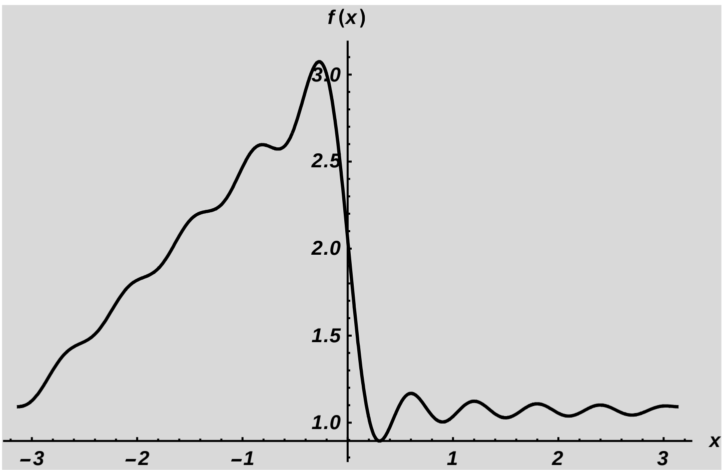
$= \frac{1}{\pi}\frac{a}{\pi}\left[x\frac{-\cos\,nx}{n} - \frac{-\sin\,nx}{n^2}\right]_{-\pi}^{0} + \;\frac{a}{\pi}\left[\frac{-\cos\,nx}{n}\right]_{-\pi}^{0}$

$= \frac{1}{\pi}\frac{a}{\pi}\left[(0 - 0) - \left(\frac{(-\pi)(-(-1)^n)}{n} - 0\right)\right] + (-\;\frac{a}{n\pi} + \frac{(-1)^n a}{n\pi}) = -\frac{a}{n\pi}$ then $b_n = -\frac{a}{n\pi}$

Then $f(x) = \frac{\boldsymbol{a}}{\boldsymbol{4}} + \sum_{n=1}^{\infty}\left(\left\{\frac{a}{\pi^2 n^2}(1 - (-1)^n)\right\}\cos nx + \{-\frac{a}{n\pi}\}\sin nx\right)$

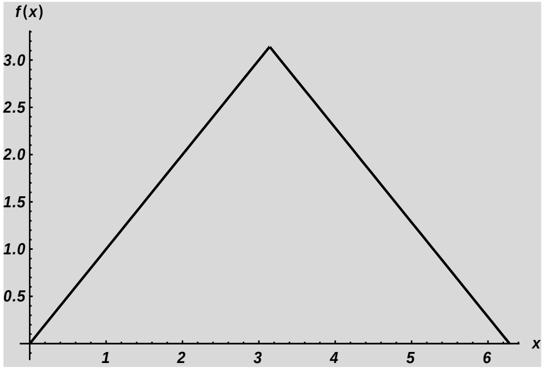
$$= \frac{\boldsymbol{a}}{\boldsymbol{4}} + \sum\nolimits_{\boldsymbol{1,3,5,7\ldots}}\frac{2a}{\pi^2 n^2}\cos nx - \sum\nolimits_{1,2,3,4}\frac{a\;\sin nx}{n\pi}$$

$$f(x) = \frac{\boldsymbol{\pi}}{\boldsymbol{2}} + \frac{2a}{\pi^2}\left(\frac{\mathbf{cos}\,x}{\mathbf{1^2}} + \frac{\mathbf{cos}\,3x}{\mathbf{3^2}} + \frac{\mathbf{cos}\,5x}{\mathbf{5^2}} + \frac{\mathbf{cos}\,7x}{\mathbf{7^2}}\ldots..\right) - \frac{a}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5}\ldots\ldots\ldots\right)$$



Q.13 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = \begin{cases} x, & for\ 0 < x < \pi \\ 2\pi - x, & for\ \ \pi < x < 2\pi \end{cases} \quad \text{where } f(x+2\pi) = f(x)$$



Then $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} x\,dx + \frac{1}{2\pi}\int_{\pi}^{2\pi}(2\pi - x)\,dx = \frac{\pi}{2}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_0^{\pi} x\ \cos nx\ dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{\pi} x\ \cos nx\ dx + 2\int_{\pi}^{2\pi}\cos nx\,dx - \frac{1}{\pi}\int_{\pi}^{2\pi} x\ \cos nx\ dx$

$= \frac{1}{\pi}\ \left[x\frac{\sin\ nx}{n} - \frac{-\cos\ nx}{n^2}\right]_0^{\pi} + 2\left[\frac{\sin\ nx}{n}\right]_{\pi}^{2\pi} - \frac{1}{\pi}\left[x\frac{\sin\ nx}{n} - \frac{-\cos\ nx}{n^2}\right]_{\pi}^{2\pi}$

$= \frac{1}{\pi}\left\{\left(0 - \frac{-(-1)^n}{n^2}\right) - \left(0 - \frac{-1}{n^2}\right)\right\}\ +0 - \frac{1}{\pi}\ \left[\left\{\left(0 - \frac{-1}{n^2}\right) - \left(0 - \frac{-(-1)^n}{n^2}\right)\right\}\right] = \frac{2}{\pi n^2}((-1)^n - 1)$ then, $a_n = \begin{cases} -\frac{4}{\pi n^2}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$

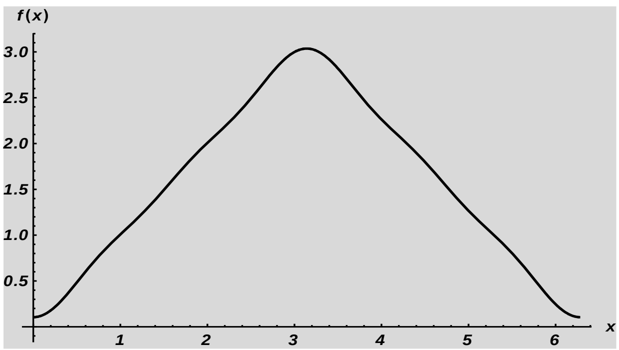
$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{\pi} x\ \sin nx\ dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx$

$= \frac{1}{\pi}\int_0^{\pi} x\ \sin nx\ dx + 2\int_{\pi}^{2\pi}\sin nx\,dx - \frac{1}{\pi}\int_{\pi}^{2\pi} x\ \sin nx\ dx$

$= \frac{1}{\pi}\ \left[x\frac{-\cos\ nx}{n} - \frac{-\sin\ nx}{n^2}\right]_0^{\pi} + 2\left[\frac{-\cos\ nx}{n}\right]_{\pi}^{2\pi} - \frac{1}{\pi}\left[x\frac{-\cos\ nx}{n} - \frac{-\sin\ nx}{n^2}\right]_{\pi}^{2\pi}$

$= \frac{1}{\pi}\left\{\left(\pi\frac{-(-1)^n}{n} - 0\right)\right\} + 2\left\{-\frac{1}{n} + \frac{(-1)^n}{n}\right\} - \frac{1}{\pi}\left\{\left(2\pi\frac{-1}{n} - \pi\frac{-(-1)^n}{n}\right)\right\} = -\frac{(-1)^n}{n} - \frac{2}{n} + \frac{2(-1)^n}{n} + \frac{2}{n} - \frac{(-1)^n}{n} = 0$

$f(x) = \frac{\pi}{2} + \sum_{1,3,5}(-\frac{4}{\pi n^2})\cos nx\ = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2}\dots..\right)$



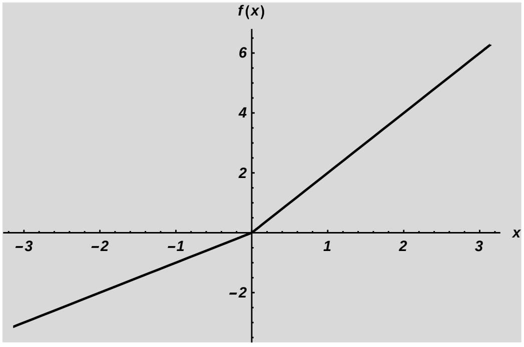
Now putting $x = 0$ we have $f(x) = 0$

Then from above equation $0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\dots..\right)$

Then $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \dots\dots\dots.(proved)$

Q.14 Find the Fourier series expansion of the function mathematically defined as

$$f(x) = \begin{cases} x, & for -\pi < x < 0 \\ 2x, & for \quad 0 < x < \pi \end{cases} \quad \text{where } f(x+2\pi) = f(x)$$



Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2x\,dx = \frac{\pi}{4}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} x\ \cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\cos nx\ dx$$

$$= \frac{1}{\pi}\ \int_{-\pi}^{0} x\cos nx\ dx + \frac{2}{\pi}\ \int_{0}^{\pi} x\cos nx\ dx\ = \frac{1}{\pi}\ \left[x\frac{\sin\ nx}{n} - \frac{-\cos\ nx}{n^2}\right]_{-\pi}^{0} + \frac{2}{\pi}\ \left[x\frac{\sin\ nx}{n} - \frac{-\cos\ nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\ \left[\left(0 - \frac{-1}{n^2}\right) - \left(0 - \frac{-(-1)^n}{n^2}\right)\right] + \frac{2}{\pi}\ \left[\left(0 - \frac{-(-1)^n}{n^2}\right) - \left(0 - \frac{-1}{n^2}\right)\right]\ 0 = \frac{1}{\pi n^2}((-1)^n - 1) \qquad \text{then, } a_n = \begin{cases} -\frac{2}{\pi n^2}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

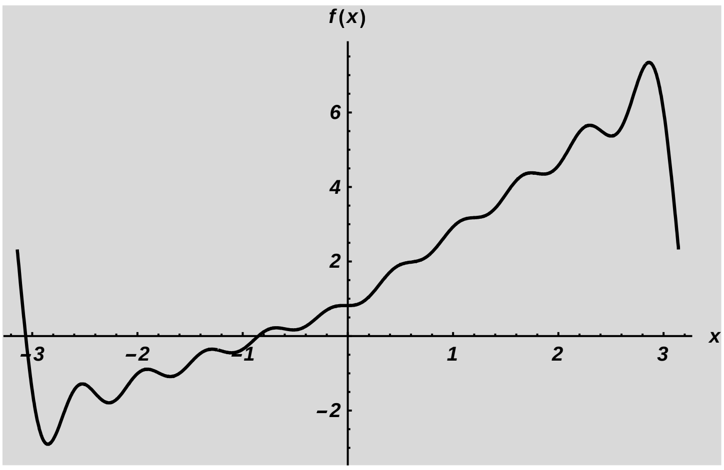
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\sin nx\ dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\ dx\ + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\ dx\ = \frac{1}{\pi}\ \left[x\frac{-\cos\ nx}{n} - \frac{-\sin\ nx}{n^2}\right]_{-\pi}^{0} + \frac{2}{\pi}\ \left[x\frac{-\cos\ nx}{n} - \frac{-\sin\ nx}{n^2}\right]_{0}^{\pi}$$

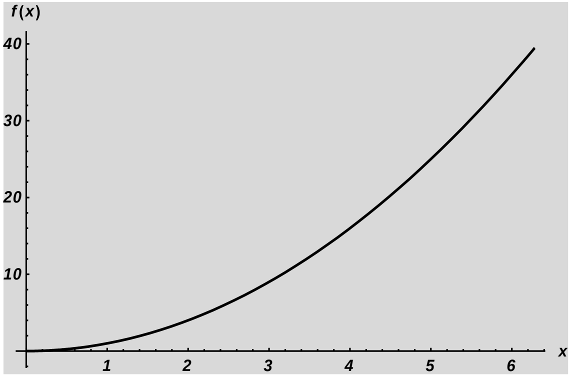
$$= \frac{1}{\pi}\left[(0-0) - \left(\frac{(-\pi)(-(-1)^n)}{n} - 0\right)\right] + \frac{2}{\pi}\left[\left(\frac{(\pi)(-(-1)^n)}{n} - 0\right) - (0-0) - \right] = -\frac{3(-1)^n}{n} \quad \text{then, } b_n = \begin{cases} \frac{3}{n}, & for\ odd\ n \\ -\ \frac{3}{n} & for\ even\ n \end{cases}$$

Then $f(x) = \frac{\pi}{4} + \sum_{1,3,5,7\ldots}(-\frac{2}{\pi n^2})\cos nx + \sum_{1,3,5\ldots}\frac{3\ \sin nx}{n} - \sum_{2,4,6\ldots}\frac{3\sin nx}{n}$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} \ldots..\right) + 3\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \ldots\ldots\ldots\right) - 3\left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} \ldots\ldots\ldots\ldots\right)$$



Q15. Expand the function $f(x) = x^2$ in the range $0 < x < 2\pi$ in Fourier series.



Then $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{2\pi}\frac{\mathbf{8\pi^3}}{\mathbf{3}} = \frac{\mathbf{4\pi^2}}{\mathbf{3}}$

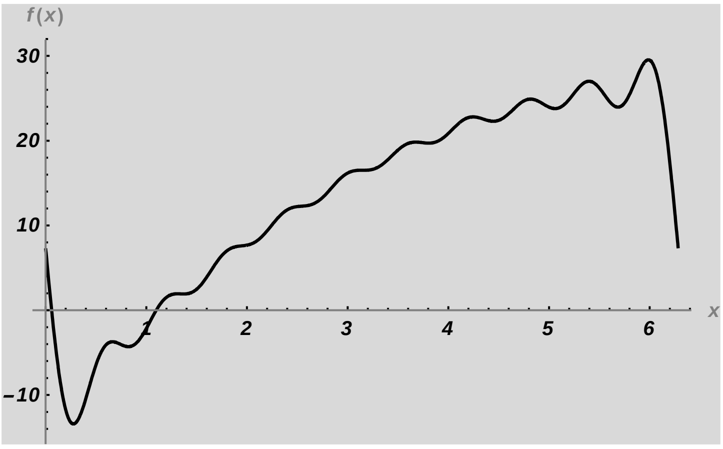
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_0^{2\pi} x^2\ \cos nx\ dx$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\frac{-\cos nx}{n^2} + 2\frac{-\sin nx}{n^3}\right]_0^{2\pi} = \frac{1}{\pi}\left[-\frac{4\pi(-1)}{n^2}\right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{2\pi} x^2\ \sin nx\ dx$$

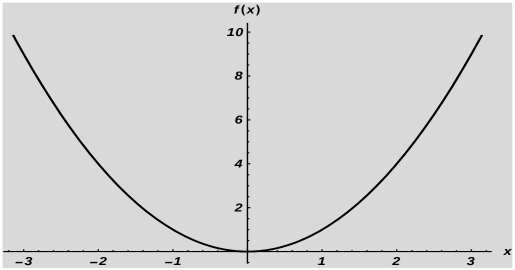
$$= \frac{1}{\pi}\left[x^2\frac{-\cos nx}{n} - 2x\frac{-\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\left(\frac{4\pi^2(-1)}{n} + \frac{2}{n^3}\right) - \frac{2}{n^3}\right] = -\frac{4\pi}{n}$$

Then $f(x) = \frac{\mathbf{4\pi^2}}{\mathbf{3}} + \sum_{n=1}^{\infty}\left[\left(\frac{4}{n^2}\right)\cos nx + \left(-\frac{4\pi}{n}\right)\sin nx\right]$



Q16. Expand the function $f(x) = x^2$ in the range $-\pi < x < \pi$ in Fourier series.

Hence show that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$+........



Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{2\pi}[\frac{\boldsymbol{\pi^3}}{\mathbf{3}} - \left(-\frac{\boldsymbol{\pi^3}}{\mathbf{3}}\right)] = \frac{\boldsymbol{\pi^2}}{\mathbf{3}}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ \cos nx\ dx$

$= \frac{1}{\pi}\ \left[x^2 \frac{\sin\ nx}{n} - 2x\frac{-\cos\ nx}{n^2} + 2\frac{-\sin\ nx}{n^3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\ \left[-2\pi\frac{-(-1)^n}{n^2} - \left(-2(-\pi)\frac{-(-1)^n}{n^2}\right)\right] = (-1)^n\ \frac{4}{n^2}$

$$a_n = \begin{cases} -\dfrac{4}{n^2}, & for\ n\ odd \\ \dfrac{4}{n^2}, & for\ n\ even \end{cases}$$

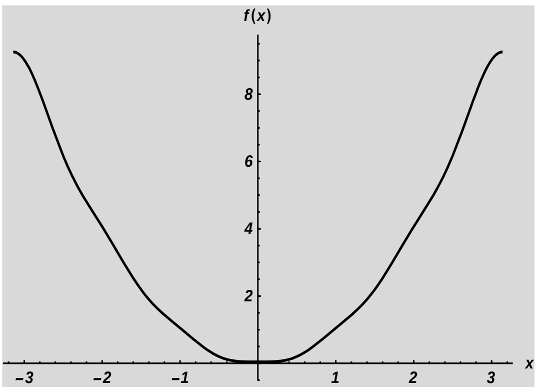
$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ \sin nx\ dx$

$= \frac{1}{\pi}\ \left[x^2 \frac{-\cos\ nx}{n} - 2x\frac{-\sin\ nx}{n^2} + 2\frac{\cos\ nx}{n^3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\ \left[\left(\pi^2\frac{-(-1)^n}{n} + \frac{2(-1)^n}{n^3}\right) - \left(\pi^2\frac{-(-1)^n}{n} + \frac{2(-1)^n}{n^3}\right)\right] = 0$

As the function is even then $\boldsymbol{b_n}$ should be 0

Then $f(x) = \frac{\boldsymbol{\pi^2}}{\mathbf{3}} + \sum_{n=1}^{\infty}\left[\left((-1)^n\ \frac{4}{n^2}\right)\cos nx + (0)\sin nx\right]$

$$= \frac{\boldsymbol{\pi^2}}{\mathbf{3}} - 4\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} \ldots..\right) + 4\ \left(\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \frac{\cos 8x}{8^2} + \cdots..\right)$$



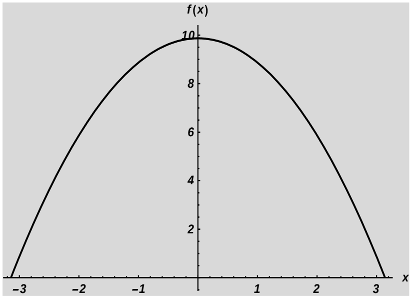
Putting $x = \pi$ we get f(π) $= \pi^2$

Then $\pi^2 = \frac{\boldsymbol{\pi^2}}{\mathbf{3}} + \mathbf{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\right)$ **then,** $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$+........

Note that $\sum_{1,2,3}\frac{1}{n^2} = \ \xi(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$+........ [where $\xi(k) = \sum\frac{1}{n^k}$ is Riemann Zeta Function]

Then $\xi(2) = \sum\frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64$

Q17. Expand the function $f(x) = \pi^2 - x^2$ in the range $-\pi < x < \pi$ in Fourier series.



As the function is even then b_n should be 0

Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 2 \times \frac{1}{2\pi}\int_0^{\pi}(\pi^2 - x^2)\, dx = \frac{1}{\pi}[\,\boldsymbol{\pi^3} - \left(\frac{\pi^3}{3}\right)] = \frac{2\,\pi^2}{3}$

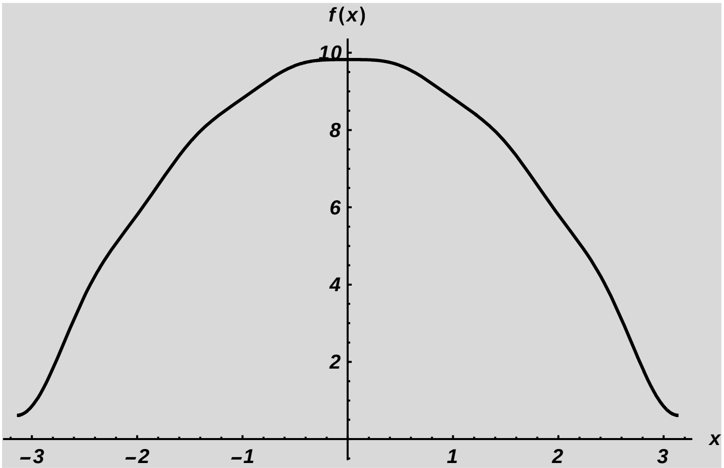
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;\, dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\;\cos nx\;\, dx$

$$= 0 - \frac{2}{\pi}\left[x^2\frac{\sin\, nx}{n} - 2x\frac{-\cos\, nx}{n^2} + 2\frac{-\sin\, nx}{n^3}\right]_0^{\pi} = -\frac{2}{\pi}\left[-2\pi\frac{-(-1)^n}{n^2} - (0)\right] = -(-1)^n\,\frac{4}{n^2}$$

$$a_n = \begin{cases} \dfrac{4}{n^2}, & for\; n\; odd \\ -\dfrac{4}{n^2}, & for\; n\; even \end{cases}$$

Then $f(x) = \frac{2\,\pi^2}{3} + \sum_{n=1}^{\infty}\left[\left(-(-1)^n\,\frac{4}{n^2}\right)\cos nx + (0)\sin nx\right]$

$$= \frac{2\,\pi^2}{3} + 4\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} \ldots ..\right) - \mathbf{4}\,\left(\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \frac{\cos 8x}{8^2} + \cdots ..\right)$$



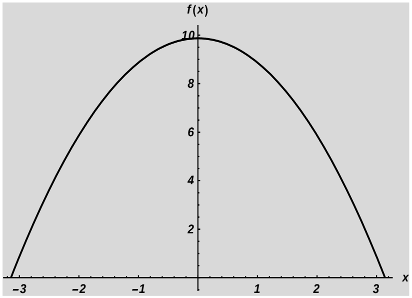
Putting $x = \pi$ we get f$(\pi) = 0$

Then $0 = \frac{2\,\pi^2}{3} - \mathbf{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\right)$ **then,** $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$+........

Note that $\sum_{1,2,3}\frac{1}{n^2} = \;\xi(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$+........ [where $\xi(k) = \sum\frac{1}{n^k}$ is Riemann Zeta Function]

Then $\xi(2) = \sum\frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64$

Q18. Expand the function $f(x) = l^2 - x^2$ in the range $-l < x < l$ in Fourier series.



As the function is even then b_n should be 0

Then $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = 2 \times \frac{1}{2l}\int_{0}^{l}(l^2 - x^2)\,dx = \frac{1}{l}[\,\boldsymbol{l^3} - \left(\frac{\boldsymbol{l^3}}{\mathbf{3}}\right)] = \frac{\mathbf{2}\,\boldsymbol{l^2}}{\mathbf{3}}$

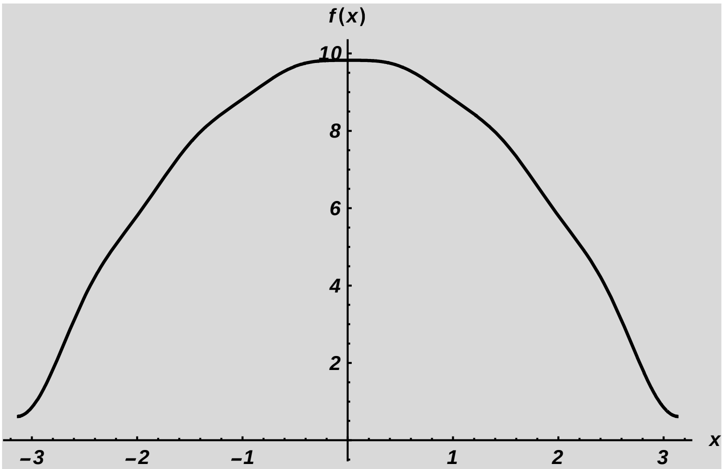
$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l}(l^2 - x^2)\,\cos\frac{n\pi x}{l}\,dx$

$$= 0 - \frac{2}{l}\left[x^2\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - 2x\frac{-\cos\frac{n\pi x}{l}}{(\frac{n\pi}{l})^2} + 2\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l = -\frac{2}{l}\left[-2l\frac{-(-1)^n}{\left(\frac{n\pi}{l}\right)^2} - (0)\right] = -(-1)^n\,\frac{4l^2}{n^2\pi^2}$$

$$a_n = \begin{cases} \dfrac{4l^2}{n^2\pi^2}, & for\ n\ odd \\ -\dfrac{4l^2}{n^2\pi^2}, & for\ n\ even \end{cases}$$

Then $f(x) = \frac{\mathbf{2}\,\boldsymbol{\pi^2}}{\mathbf{3}} + \sum_{n=1}^{\infty}\left[\left(-(-1)^n\,\frac{4l^2}{n^2\pi^2}\right)\cos\frac{n\pi x}{l} + (0)\sin\frac{n\pi x}{l}\right]$

$$= \frac{\mathbf{2}\,\boldsymbol{\pi^2}}{\mathbf{3}} + \frac{4l^2}{\pi^2}\left(\frac{\mathbf{cos}\,\boldsymbol{x}\frac{\pi}{l}}{\mathbf{1^2}} + \frac{\mathbf{cos\,3}\boldsymbol{x}\frac{\pi}{l}}{\mathbf{3^2}} + \frac{\mathbf{cos\,5}\boldsymbol{x}\frac{\pi}{l}}{\mathbf{5^2}} + \frac{\mathbf{cos\,7}\boldsymbol{x}\frac{\pi}{l}}{\mathbf{7^2}} \ldots..\right) - \frac{4l^2}{\pi^2}\left(\frac{\mathbf{cos\,2}\boldsymbol{x}\frac{\pi}{l}}{\mathbf{2^2}} + \frac{\mathbf{cos\,4}\boldsymbol{x}\frac{\pi}{l}}{\mathbf{4^2}} + \frac{\mathbf{cos\,6}\boldsymbol{x}\frac{\pi}{l}}{\mathbf{6^2}} + \frac{\mathbf{cos\,8}\boldsymbol{x}\frac{\pi}{l}}{\mathbf{8^2}} + \cdots..\right)$$



Putting $x = \pi$ we get $\mathrm{f}(\pi) = 0$

Then $0 = \frac{\mathbf{2}\,\boldsymbol{\pi^2}}{\mathbf{3}} - \mathbf{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\right)$ **then,** $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$+........

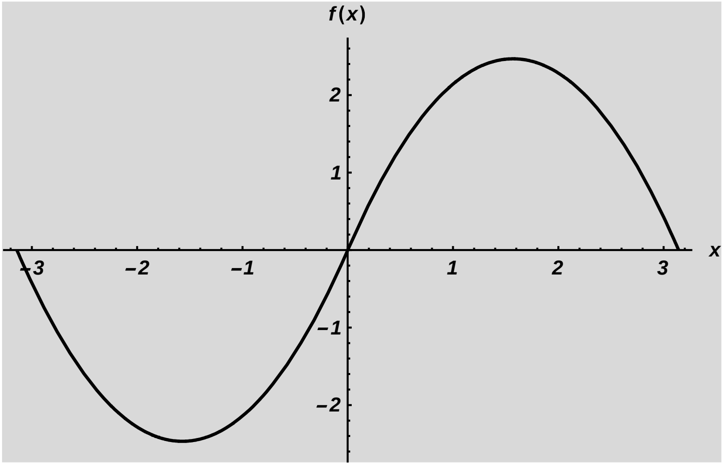
Note that $\sum_{1,2,3}\frac{1}{n^2} = \xi(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$+........ [where $\xi(k) = \sum\frac{1}{n^k}$ is Riemann Zeta Function]

Then $\xi(2) = \sum\frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64$

Q.19. Find the Fourier series expansion of the function (square wave) mathematically defined as

$$f(x) = \begin{cases} x(\pi + x), & for - \pi < x < 0 \\ x(\pi - x), & for \quad 0 < x < \pi \end{cases} \quad \text{where } f(x + 2\pi) = f(x)$$

Hence show that Then $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$..



As the function is odd then a_0, a_n should be 0

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx \; dx$$

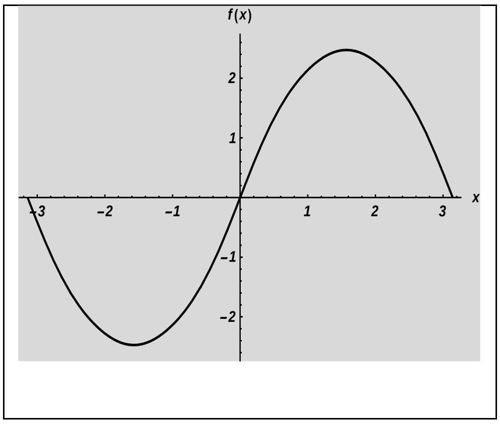
$$= \frac{1}{\pi}\int_{-\pi}^{0} \boldsymbol{x(\pi + x)}\sin nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} \boldsymbol{x(\pi - x)}\sin nx \; dx \ldots\ldots(\text{i})$$

Replacing $x = -x$ in the first integral in equatiobn (i)we get

$$b_n = \frac{1}{\pi}\int_{\pi}^{0} -\boldsymbol{x(\pi - x)}\sin(-nx)\;(-dx) + \frac{1}{\pi}\int_{0}^{\pi} \boldsymbol{x(\pi - x)}\sin nx \; dx$$

$$=2\times\frac{1}{\pi}\int_{0}^{\pi} \boldsymbol{x(\pi - x)}\sin nx \; dx = 2\int_{0}^{\pi} \boldsymbol{x}\sin nx \; dx - \frac{2}{\pi}\int_{0}^{\pi} \boldsymbol{x^2}\sin nx \; dx$$

$$= -\frac{2(-1)^n\pi}{n} - \frac{2}{\pi}\left\{\frac{1}{n}\left(\frac{(-1)^n 2}{n^2} - \pi^2(-1)^n\right) - \frac{2}{n^3}\right\} = \frac{4}{\pi n^3}(1 - (-1)^n) \quad \text{then } b_n = \begin{cases} \frac{8}{\pi n^3}, & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$



$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1 - (-1)^n)[\sin nx]$$

$$f(x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \frac{\sin 7x}{7^3} + \cdots \ldots\ldots\ldots\right)$$

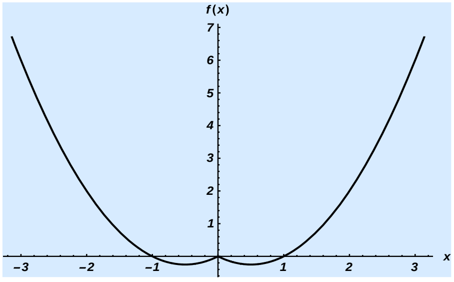
Putting $x = \pi/2$ we get $f(\pi/2) = \frac{\pi^2}{4}$

Then from above equation $\frac{\pi^2}{4} = \frac{8}{\pi}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots ..\right)$

Hence $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$..

Q.20 Find the Fourier series expansion of the function (square wave) mathematically defined as

$$f(x) = \begin{cases} x^2 + x, & for -\pi < x < 0 \\ x^2 - x, & for \quad 0 < x < \pi \end{cases} \quad \text{where } f(x+2\pi) = f(x)$$



As the function is even then b_n should be 0

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0}(\boldsymbol{x^2+x})\,dx + \frac{1}{2\pi}\int_{0}^{\pi}(\boldsymbol{x^2-x})\,dx = \frac{(2\pi^3-3\pi^2)}{6\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0}(\boldsymbol{x^2+x})\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\boldsymbol{x^2-x})\cos nx\,dx \ldots\ldots$(i)

Replacing $x = -x$ in the first integral in equatiobn (i)we get

$a_n = \frac{1}{\pi}\int_{\pi}^{0}(\boldsymbol{x^2-x})\cos(-nx)\;(-dx) + \frac{1}{\pi}\int_{0}^{\pi}(\boldsymbol{x^2-x})\cos nx\,dx$

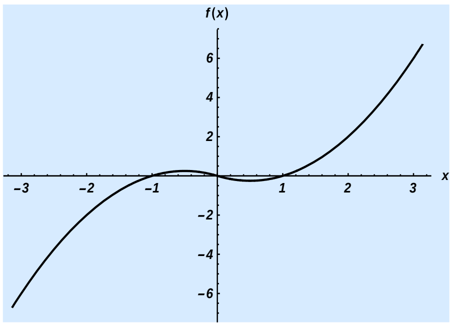
$=2\times\frac{1}{\pi}\int_{0}^{\pi}(\boldsymbol{x^2-x})\cos nx\,dx = 2\times\frac{1}{\pi}\int_{0}^{\pi}\boldsymbol{x^2}\cos nx\,dx - 2\times\frac{1}{\pi}\int_{0}^{\pi}\boldsymbol{x}\cos nx\,dx$

$= 2\times\frac{1}{\pi}\times\frac{2\pi(-1)^n}{n^2} - 2\times\frac{1}{\pi}\times\frac{1}{n^2}((-1)^n-1) = \frac{4(-1)^n}{n^2} - \frac{2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2}$

$f(x) = \frac{(2\pi^3-3\pi^2)}{6\pi} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2} - \frac{2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2}\right)[\cos nx]$

Q.21. Find the Fourier series expansion of the function (square wave) mathematically defined as

$$f(x) = \begin{cases} -x^2 - x, & for -\pi < x < 0 \\ x^2 - x, & for \quad 0 < x < \pi \end{cases} \quad \text{where } f(x+2\pi) = f(x)$$



As the function is odd then a_0, $\boldsymbol{a_n}$ should be 0

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0}(\boldsymbol{-x^2-x})\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\boldsymbol{x^2-x})\sin nx\,dx \ldots\ldots$(i)

Replacing $x = -x$ in the first integral in equatiobn (i)we get

$b_n = \frac{1}{\pi}\int_{\pi}^{0}(\boldsymbol{-x^2+x})\sin(-nx)\;(-dx) + \frac{1}{\pi}\int_{0}^{\pi}(\boldsymbol{x^2-x})\sin nx\,dx$

$=2\times\frac{1}{\pi}\int_{0}^{\pi}(\boldsymbol{x^2-x})\sin nx\,dx = 2\times\frac{1}{\pi}\int_{0}^{\pi}\boldsymbol{x^2}\sin nx\,dx - \frac{2}{\pi}\int_{0}^{\pi}\boldsymbol{x}\sin nx\,dx \;= \frac{2(2(-1)^n-\pi^2(-1)^n n^2-2)}{\pi n^3} + \frac{2(-1)^n}{n}$

$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2(2(-1)^n-\pi^2(-1)^n n^2-2)}{\pi n^3} + \frac{2(-1)^n}{n}\right\}[\sin nx]$