Vector Calculus (27 Lectures)

Recapitulation of vectors: Properties of vectors under rotations. Scalar product and its invariance under rotations. Vector product, Scalar triple product and their interpretation in terms of area and volume respectively. Scalar and Vector fields. Vector Differentiation: Directional derivatives and normal derivative. Gradient of a scalar field and its geometrical interpretation. Divergence and curl of a vector field. Del and Laplacian operators. Vector identities using Kronecker delta and Levi-civita symbols. Vector Integration: Ordinary Integrals of Vectors. Multiple integrals, Jacobian. Notion of infinitesimal line, surface and volume elements. Line, surface and volume integrals of Vector fields. Flux of a vector field. Gauss' divergence theorem, Green's and Stokes Theorems and their applications (no rigorous proofs).

Recommended Books-

1) "Vector Analysis" by Murray R. Spiegel (Schaum's Outline Series)
2) "Div, Grad, Curl, and All That: An Informal Text on Vector Calculus" by H. M. Schey
3) "Vector Calculus" by Jerrold E. Marsden and Anthony J. Tromba
4) "Mathematical Methods for Physics and Engineering" by K. F. Riley, M. P. Hobson, and S. J. Bence
5) "Mathematical Methods for Physicists" by George B. Arfken, Hans J. Weber, and Frank E. Harris.

Recapitulation of vector algebra HOME WORK

Kronecker delta

In mathematics, the Kronecker delta (named after Leopold Kronecker) is a function of two variables,

Usually just non-negative integers. The function is **1** if the variables are equal, and **0** otherwise.

$$\delta_{ij} = \begin{cases} 1 & for\ i = j \\ 0 & for\ i \neq j \end{cases}$$

Levi-Civita symbol

In two dimensions, the **Levi-Civita** symbol is defined by:

$$\epsilon_{ij} = \begin{cases} 1 & if(i,j) = (1,2) \\ -1 & if(i,j) = (1,2) \\ 0 & if\ i = j \end{cases}$$

In three dimensions, the **Levi-Civita** symbol is defined by:

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } (i.j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i.j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if there is a repeated index} \end{cases}$$

One useful identity is $\epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

Linear dependence and independence, completeness, basis and representation of vectors. N vectors $\vec{X}_1, \vec{X}_2, \dots . \vec{X}_N,$ are called **linearly dependent** if there exists scalars $a_1, a_2, \dots \dots a_N$ not for all zeros such that $\sum a_N \vec{X}_1 = 0$. If no such scalars exit, then the vectors are said to be linearly independent.

For three vectors $\begin{matrix} \vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} \\ \vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k} \\ \vec{c} = c_x\hat{\imath} + c_y\hat{\jmath} + c_z\hat{k} \end{matrix}$ be **linearly dependent** if $\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0$

And **linearly independent** if $\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \neq 0$

i.e. this means that when vectors are linearly independent, none of the vectors can be obtained as a combination of the others.

In mathematics, a set of elements (vectors) in a vector space *V* is called a **basis**, or a set of **basis vectors**, if the vectors are linearly independent and every vector in the vector space is a linear combination of this set. In more general terms, a basis is a linearly independent spanning set.

Given a basis of a vector space *V*, every element of *V* can be expressed uniquely as a linear combination of basis vectors, whose coefficients are referred to as vector **coordinates** or **components**. A vector space can have several distinct sets of basis vectors; however each such set has the same number of elements, with this number being the dimension of the vector space.

Note that

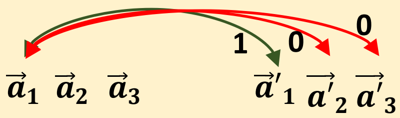
1. If there exists N linearly independent vectors, the vector space is said to be N-dimensional.
2. Consider a linear vector space V with a basis set of N linearly independent vectors, $\vec{x}_1$,$\vec{x}_2$ …$\vec{x}_3$

Any general vector $\vec{A}$ can be expressed as a linear sum of base vectors $\hat{x}_1, \vec{x}_2 \ldots \vec{x}_3$ with scalar coefficients a_i : That means any vector ($\vec{A}$) in N dimensional vector space V, can be expressed as $\vec{A} = \sum_{i=1}^{N} a_i \vec{x}_i$

3. The base vectors are said to form a complete set for that space (i.e., any vector of the space can be uniquely expressed as a linear combination of these vectors).
4. Base vectors for an N-dimensional space are not unique - any set of N linear independent vectors can form a basis.

Reciprocal Vectors Two sets of three non-coplanar vectors $(\vec{a}, \vec{b}, \vec{c})$ and $(\vec{a'}, \vec{b'}, \vec{c'})$ are called reciprocal sets if

$$\begin{array}{lll} \vec{a}.\vec{a'} = 1 & \vec{a}.\vec{b'} = 0 & \vec{a}.\vec{c'} = 0 \\ \vec{b}.\vec{a'} = 0 & \vec{b}.\vec{b'} = 1 & \vec{b}.\vec{c'} = 0 \\ \vec{c}.\vec{a'} = 0 & \vec{c}.\vec{b'} = 0 & \vec{c}.\vec{c'} = 1 \end{array}$$



Representation of $(\vec{a'}, \vec{b'}, \vec{c'})$ in terms of $(\vec{a}, \vec{b}, \vec{c})$.

Since from definition $\vec{b}.\vec{a'} = 0$ & $\vec{c}.\vec{a'} = 0$, $\vec{a'}$ is perpendicular to both $\vec{b}$ & $\vec{c}$, hence parallel to $(\vec{b} \times \vec{c})$

Then $\vec{a'} = m\,(\vec{b} \times \vec{c})$. Again $\vec{a}.\,\vec{a'} = 1$ then $\vec{a}.m\,(\vec{b} \times \vec{c}) = 1$ or, $m = \frac{1}{\vec{a}.(\vec{b} \times \vec{c})}$

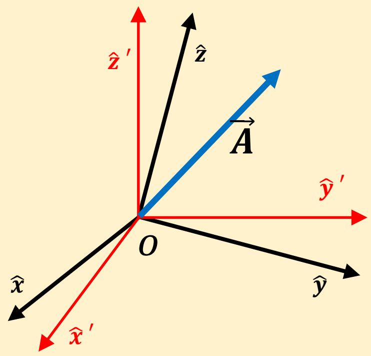
then, $\vec{a'} = \frac{(\vec{b} \times \vec{c})}{\vec{a}.(\vec{b} \times \vec{c})}$.**Similarly** $\vec{b'} = \frac{(\vec{c} \times \vec{a})}{\vec{a}.(\vec{b} \times \vec{c})}$ **and** $\vec{c'} = \frac{(\vec{a} \times \vec{b})}{\vec{a}.(\vec{b} \times \vec{c})}$

Vector and transformation. We have two equivalent ways of defining a vector: geometrically as a quantity with a magnitude and direction in space, or algebraically as a set of three numbers (in a 3D space) which we call its components. However these two ways of defining a vector are not completely equivalent, for the algebraic definition requires that a co-ordinate systems be set up, whereas the geometric definition goes without it this flaw can be remediate by making the algebraic definition also independent of any particular set of axes- which is embodied in the following equality (the same vector $\vec{A}$ is expressed in two orthogonal Cartesian co-ordinate system-primed and unprimed):

$$\vec{A} = \hat{x}'A'_x + \hat{y}'A'_y + \hat{z}'A'_z = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \;\ldots\ldots\ldots\ldots(\text{i})$$

[Where A_x, A_y&A_z are the component of $\vec{A}$ in unprimed co-ordinate system, A'_x, A'_y& A'_z Are the component of $\vec{A}$ in unprimed co-ordinate system.]

Now from equation (i)



$$\left.\begin{aligned} A'_x &= \hat{x}'.\vec{A} = (\hat{x}'.\hat{x})A_x + (\hat{x}'.\hat{y})A_y + (\hat{x}'.\hat{z})A_z \\ A'_y &= \hat{y}'.\vec{A} = (\hat{y}'.\hat{x})A_x + (\hat{y}'.\hat{y})A_y + (\hat{y}'.\hat{z})A_z \\ A'_z &= \hat{z}'.\vec{A} = (\hat{z}'.\hat{x})A_x + (\hat{z}'.\hat{y})A_y + (\hat{z}'.\hat{z})A_z \end{aligned}\right\} \text{.......(ii)}$$

$$\begin{pmatrix} A'_x \\ A'_x \\ A'_x \end{pmatrix} = \begin{pmatrix} (\hat{x}'.\hat{x}) & (\hat{x}'.\hat{y}) & (\hat{x}'.\hat{z}) \\ (\hat{y}'.\hat{x}) & (\hat{y}'.\hat{y}) & (\hat{y}'.\hat{z}) \\ (\hat{z}'.\hat{x}) & (\hat{z}'.\hat{y}) & (\hat{z}'.\hat{z}) \end{pmatrix} \begin{pmatrix} A_x \\ A_x \\ A_x \end{pmatrix}$$

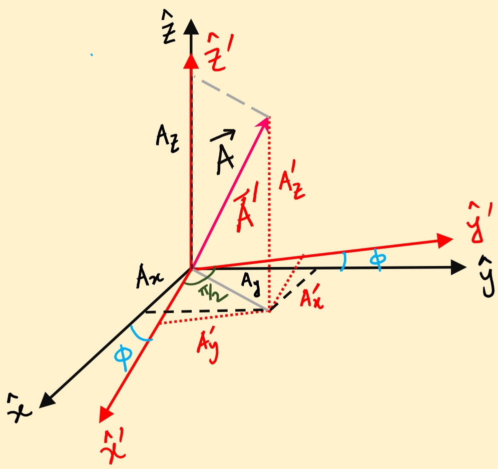
$(\hat{x}'.\hat{x}), (\hat{x}'.\hat{y})\&(\hat{x}'.\hat{z})$ **are the direction cosine** of $\hat{x}'$ axis w.r.t X,Y,Z respectively etc. The co-efficient $(\hat{x}'.\hat{x})$, $(\hat{x}'.\hat{y})$ &$(\hat{x}'.\hat{z})$ are numbers which are determined by the given co-ordinate rotation, they don't depend on $\vec{A}$.

<u>So in the process of establishing the equivalency of two definitions</u> we have ultimate reach the transformation equation (ii) for components of vector.

A vector in 3-D <u>is a set of three numbers</u> which transforms under a rotation of co-ordinate system according to equation (ii).

Note that general displacement of co-ordinate system composed of translation as well as rotation. The reason that we referred only to rotation because translation has no effect on components.

<u>A special case (ROTATION ABOUT Z-AXIS):</u>



From equⁿ (i) $A'_x = (\cos\phi)\, A_x + \{os(\frac{\pi}{2} - \phi)\}\, A_y + (\cos\frac{\pi}{2})\, A_z$

$$A'_y = \left\{\cos(\tfrac{\pi}{2} + \phi)\right\} A_x + (\cos\phi)\, A_y + (\cos\tfrac{\pi}{2})\, A_z$$

$$A'_z = (\cos\tfrac{\pi}{2})\, A_x + (\cos\tfrac{\pi}{2})\, A_y + (\cos\ 0)\, A_z$$

Then $A'_x = (\cos\phi)\ \ A_x\ + (\sin\phi)\, A_y + (0)A_z$

$$A'_y = (-\sin\phi)\, A_x + (\cos\phi)\, A_y\ + (0\,)A_z$$

$$A'_z = \ (\ 0\ \ )\ A_x + (\ \ 0\ \ )A_y +\ (1\,)A_z$$

We have then, $\begin{pmatrix} A'_x \\ A'_x \\ A'_x \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_x \\ A_x \end{pmatrix}$

<u>Orthogonal transformation.</u> In the above transformation the **connection between the directions cosines** arise from the fact that the basis vectors are orthogonal to each other and have unit magnitude in the both co-ordinate system. The following change of symbol is very useful to study the properties of the nine direction cosine in brief notation. With $x \rightarrow 1, y \rightarrow 2, z \rightarrow 3.$

$$A'_1 = \alpha_{11}A_1 + \alpha_{12}A_2 + \alpha_{13}A_3$$

$$A'_2 = \alpha_{21}A_1 + \alpha_{22}A_2 + \alpha_{33}A_3 \quad \text{.............(iii)}$$

$$A'_3 = \alpha_{31}A_1 + \alpha_{32}A_2 + \alpha_{33}A_3$$

or, $A'_i = \sum_{j=1}^{3} \alpha_{ij}A_j$ **where** $i = 1,2,3$ [As here j is repeated (dummy) index then $A'_i = \alpha_{ij}A_j$]

Since the magnitude must be same in the both system then

$$\sum_{i=1}^{3} {A'_i}^2 = \sum_{i=1}^{3} A_i^2 \text{.................(iv)}$$

Then $\sum_{i=1}^{3}\left(\sum_{j=1}^{3} \alpha_{ij}A_j\right)\left(\sum_{k=1}^{3} \alpha_{ik}A_k\right) = \sum_{i=1}^{3} A_i^2$ **Or,** $\sum_{j,k=1}^{3}\left(\sum_{i=1}^{3} \alpha_{ij}\alpha_{ik}\right) A_jA_k = \sum_{i=1}^{3} A_i^2$

this equation will reduced to equation **(iv) if** $\sum_{i=1}^{3} \alpha_{ij}\alpha_{ik} = \begin{cases} 1 \; for \; j = k \\ 0 \; for \; j \neq k \end{cases}$

$$\text{or, } \sum_{i=1}^{3} \alpha_{ij}\alpha_{ik} = \delta_{jk} \quad j,k = 1,2,3 \text{..........(v)}$$

[That means **in matrix notation** $\alpha^T\alpha = \mathbb{I}$ where α^T is the **transpose** of the matrix α $\mathbb{I}$= **unit matrix Then** $|\alpha^T|.|\alpha| = 1$ **or** $|\alpha| = \pm 1$]

Equation (iv) formally express six equation which **reduced** the independent quantities (α_{ij}) **from nine to three.**

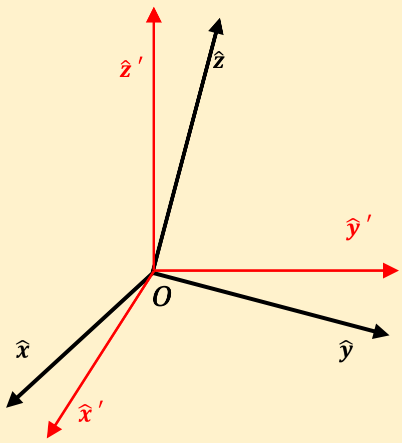
Any linear transformation ($A'_i = \sum_{j=1}^{3} \alpha_{ij}A_j$ **where** $i = 1,2,3$) **which has the property** $\sum_{i=1}^{3} \alpha_{ij}\alpha_{ik} = \delta_{jk}$ $j,k = 1,2,3$ **is called orthogonal transformation and equation (iv) is known as orthogonal condition.**

<u>Note that in terms of unit vector:</u>

$$\hat{x}'_1 = \alpha_{11}\hat{x}_1 + \alpha_{12}\hat{y}_2 + \alpha_{13}\hat{z}_3$$

$$\hat{x}'_2 = \alpha_{21}\hat{x}_1 + \alpha_{22}\hat{y}_2 + \alpha_{33}\hat{z}_3 \quad \text{.............(vi)}$$

$$\hat{x}'_3 = \alpha_{31}\hat{x}_1 + \alpha_{32}\hat{y}_2 + \alpha_{33}\hat{z}_3$$



For orthogonal set.$[\hat{x}'_1, \hat{x}'_2, \hat{x}'_3] = [\hat{x}_1, \hat{x}_2, \hat{x}_3]$

$$= \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \pm 1$$

Or in general $[\hat{x}_i, \hat{x}_j, \hat{x}_k] = \begin{vmatrix} \alpha_{i1} & \alpha_{i2} & \alpha_{i3} \\ \alpha_{j1} & \alpha_{j2} & \alpha_{j3} \\ \alpha_{k1} & \alpha_{k2} & \alpha_{k3} \end{vmatrix} = \varepsilon_{ijk}$ $i,j,k = 1,2,3$

In similarly with vector transformation a tensor T is defined as of N[th] rank in 3-D space may be defined as a quantity of 3^N component which transformed under orthogonal transformation of the co-ordinate according to the following scheme:

$$T'_{ijk\ldots\ldots} = \sum_{lmn\ldots\ldots} \alpha_{il}\alpha_{jm}\alpha_{kn} \ldots\ldots.. T_{ijk\ldots\ldots}$$

IN THREE DIMENSION.

A scalar is a zero rank tensor has $3^0 = 1$ component, which is invariant $(A' = A)$under orthogonal transformation of the co-ordinate. A vector is a one rank tensor has $3^1 = 3$ components, which transforms as $A'_i = \sum_{j=1}^{3} \alpha_{ij} A_j$ under orthogonal transformation.

A second rank tensor has $3^2 = 9$ components, which transforms as

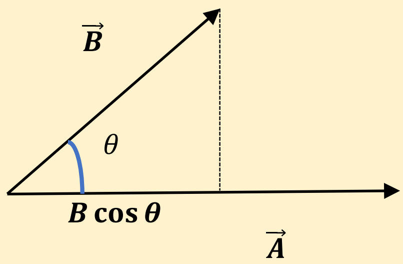
$T'_{ij} = \sum_{lmn......} \alpha_{il}\alpha_{jm} T_{ij}$ under orthogonal transformation of the co-ordinate.

QUANTITIES→ SCALERS +NON-SCALES (VECTORS, LINEAR VECTOR FUNCTIONS, DYADIC, TENSORS)

Multiplication of vector:

Multiplication by a scalar: $c\vec{A} = \vec{A}c$

Scalar product or dot product of two vectors:



The Scalar product or dot product of two vectors $\vec{A}$ & $\vec{B}$ symbolically represented as $\vec{A}.\vec{B}=|A||B|\cos\theta$

We may define $\vec{A}.\vec{B}=|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}\cot^2\theta$ where θ is the angle between $\vec{A}$ & $\vec{B}$. But there is no use of this type definition in physics

Geometrical interpretation: The scalar product of two vectors is the product of magnitude of the first vector and the projection of the second vector in the direction of the first.

EXAM: **Work done** is the scalar product of force and displacement as by definition work done is the product of displacement and component of force along displacement.

Properties of DOT product

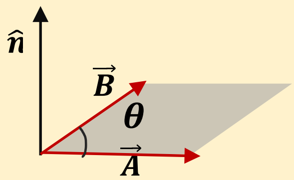
1. **Commutative:** $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
2. **Distributive:** $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$
3. **Non-Associative:** Since the dot product between a scalar and a vector is not allowed.
4. **Scalar Multiplication:** $(k\vec{A}).\vec{B} = k(\vec{A}.\vec{B}) = \vec{A}.(k\vec{B})$
5. **Orthogonality:** If $\vec{A}.\vec{B} = 0$ then the vectors are orthogonal (perpendicular) to each other.
6. **Non-negativity:** $\vec{A}.\vec{A} \geq 0$

Vector product or cross product of two vector:

The cross product of two vectors $\vec{A}$ & $\vec{B}$ symbolically represented as $\vec{A} \times \vec{B}=\{|A||B|\sin\theta\}(\hat{n})$

where θ is the angle between $\vec{A}$ & $\vec{B}$. And is the unit normal vector to the plane containing $\vec{A}$ & $\vec{B}$

Geometrical interpretation: The vector product of two vectors represent the vector area



of the Parallelogram whose adjacent sides are $\vec{A}$ & $\vec{B}$

proof: $\vec{A} \times \vec{B} = |A||B|\sin\theta\}(\hat{n})$

$= (base \times height)(\hat{n})$

$=$ **Vector area of parallelogram ABCD**

EXAM: $\vec{v} = \vec{\omega} \times \vec{r}$, $\vec{L} = \vec{r} \times \vec{P}$, $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$= radius vector, $\vec{\omega}$ =angular velocity, $\vec{v}$ = linear velocity, $\vec{P}$ =Linear momentum, $\vec{L}$ = Angular momentum $\vec{F}$ =force , $\vec{\tau}$ = torque.

Properties of CROSS product:

1. Anti-Commutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

2. Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

3. Non-associativity: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

4. Scalar Multiplication: $(k\vec{A}) \times \vec{B} = k(\vec{A} \times \vec{B}) = \vec{A} \times (k\vec{B})$

5. **Orthogonality:** $\vec{A} \times \vec{B}$ **is perpendicular to both** $\vec{A}$ **and** $\vec{B}$

5. Zero vector: $\vec{A} \times \vec{B} = \vec{0}$ **if and only if a and b are parallel (or one is the zero vector).**

Scalar triple product.

For three vector $\vec{A}, \vec{B}$ &$\vec{C}$ scalar triple product is written as$\vec{A}.(\vec{B} \times \vec{C})$. This product is also known as box product and written as $[\vec{A}\vec{B}\vec{C}]$ If $\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z$ $\vec{B} = \hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z$ $\vec{C} = \hat{\imath}C_x + \hat{\jmath}C_y + \hat{k}C_z$ then

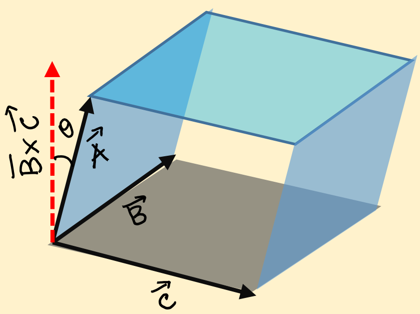
$$[\vec{A}\vec{B}\vec{C}] = \vec{A}.(\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Geometrical interpretation: the scalar triple product of three vectors represent the volume of the Parallelepiped whose adjacent edges are $\vec{A}, \vec{B}$ &$\vec{C}$

EXAM: $v_n = \hat{n}.\vec{v} = \hat{n}.(\vec{\omega} \times \vec{r})$,

$L_n = \hat{n}.\vec{L} = \hat{n}.(\vec{r} \times \vec{P})$, $\tau_n = \hat{n}.\vec{\tau} = \hat{n}.(\vec{r} \times \vec{F})$

Where v_n, L_n, τ_n are component of linear vector, angular momentum and torque along $\hat{n}$.

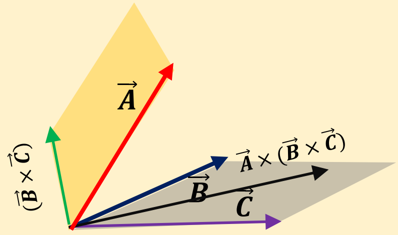


Properties of scalar triple product: $\vec{A}.(\vec{B} \times \vec{C}) = -\vec{B}.(\vec{A} \times \vec{C}) = \vec{C}.(\vec{A} \times \vec{B})$

<u>Vector triple product</u>: For three vector $\vec{A},\vec{B}$ &$\vec{C}$vector triple product is written as $\vec{A}\times(\vec{B}\ \times\vec{C})$.

This product is express as $\vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}\ \times\vec{\boldsymbol{C}})=\vec{\boldsymbol{B}}(\vec{\boldsymbol{A}}.\vec{\boldsymbol{C}})-\vec{\boldsymbol{C}}(\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}})$

<u>Geometrical interpretation:</u> $\vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}\ \times\vec{\boldsymbol{C}})$ **is a vector in a plane containing** $(\vec{B}$& $\vec{C})$**.**



As $(\vec{B}\ \times\vec{C})$ is perpendicular to **plane containing** $(\vec{B}$& $\vec{C})$

then $\vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}\ \times\vec{\boldsymbol{C}})$ **is perpendicular to both** $\vec{\boldsymbol{A}}$ **and** $(\vec{\boldsymbol{B}}\ \times\vec{\boldsymbol{C}})$

and hence lie on the plane containing $(\vec{B}$& $\vec{C})$**.**

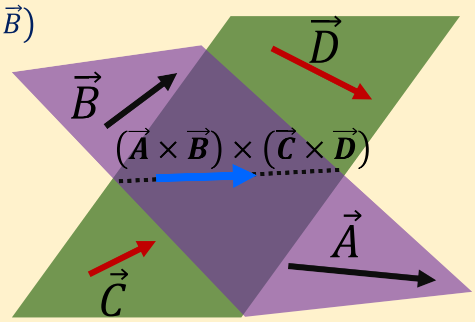
Properties of vector triple product: $\vec{A}\times(\vec{B}\ \times\vec{C})=-\vec{A}\times(\vec{C}\times\vec{B})$

<u>Vector product of four vector:</u>

$$(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}})\times(\vec{\boldsymbol{C}}\times\vec{\boldsymbol{D}})=\vec{R}\times(\vec{C}\ \times\vec{D})=\vec{C}(\vec{R}.\vec{D})-\vec{D}(\vec{R}.\vec{C})$$

$$=\vec{C}\left((\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}).\vec{D}\right)-\vec{D}\left((\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}).\vec{C}\right)$$

$$=\vec{C}[\vec{\boldsymbol{A}}\vec{\boldsymbol{B}}\vec{D}]-\vec{D}[\vec{A}\vec{B}\vec{C}]$$



<u>Geometrical interpretation</u>$(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}})\times(\vec{\boldsymbol{C}}\times\vec{\boldsymbol{D}})$ **is a vector parallel to the line of intersection of the plane containing** $(\vec{A}$& $\vec{B})$ **and the plane containing** $(\vec{\boldsymbol{C}}$&$\vec{\boldsymbol{D}})$

Repeated-index notation: $\sum_i A_iB_i=A_iB_i$ Einstein summation convention

$$\equiv A_1B_1+A_2B_2+A_3B_3$$

Dot product: $\vec{A}\cdot\vec{B}=\ \sum_i\sum_j A_iB_j\delta_{ij}=A_iB_j\delta_{ij}=A_iB_i$

Cross product: i^th component of the vector $\vec{A}\times\vec{B}$ i.e., $(\vec{A}\times\vec{B})_i=\epsilon_{ijk}A_jB_k$

Show that $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$

Proof : Using ε_{ijk} we can write index expressions for the cross product. The i^{th} component of the cross product is given by $[\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}]_i=\varepsilon_{ijk}A_jB_k$

as we check by simply writing out the sums for each value of i,

$$[\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}]_i=\varepsilon_{ijk}A_jB_k=\varepsilon_{xyz}A_yB_z+\varepsilon_{xzy}A_zB_y+(\text{ all other terms are zero })=A_yB_z-A_zB_y$$

$$[\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}]_j=\varepsilon_{jki}A_kB_i=\varepsilon_{yzx}A_zB_x+\varepsilon_{yxz}A_xB_z+(\text{ all other terms are zero })=A_zB_x-A_xB_z$$

$$[\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}]_k=\varepsilon_{kij}A_iB_j=\varepsilon_{zxy}A_xB_y+\varepsilon_{zyx}A_yB_x+(\text{ all other terms are zero })=A_xB_y-A_yB_x$$

Hence $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}=\hat{\boldsymbol{\imath}}\,(A_yB_z-A_zB_y)+\hat{\boldsymbol{\jmath}}\,(A_zB_x-A_xB_z)+\hat{\boldsymbol{k}}(A_xB_y-A_yB_x)=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$

Show that $\vec{A}.(\vec{B}\times\vec{C}) = -\vec{B}.(\vec{A}\times\vec{C}) = \vec{C}.(\vec{A}\times\vec{B})$

Proof : In Levi-Civita symbol language this is:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i(\vec{B}\times\vec{C})_i \qquad \text{[using Einstein summation convention]}$$

$$= A_i\epsilon_{ijk}B_jC_k = B_j\epsilon_{jki}C_kA_i = C_k\epsilon_{kij}A_iB_j$$

$$= -A_i\epsilon_{ikj}C_kB_j = -B_j\epsilon_{jik}A_iC_k = -C_k\epsilon_{kji}B_jA_i$$

Hence $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = -\vec{A}\cdot(\vec{C}\times\vec{B}) = -\vec{B}\cdot(\vec{A}\times\vec{C}) = -\vec{C}\cdot(\vec{B}\times\vec{A})$

Show That $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

To derive some identities, again let's use the Levi- Civita symbol:

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk}A_j(\vec{B}\times\vec{C})_k = \epsilon_{ijk}A_j\epsilon_{kmn}B_mC_n = \epsilon_{kij}\epsilon_{kmn}A_jB_mC_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})A_jB_mC_n$$

$$= \delta_{im}\delta_{jn}A_jB_mC_n - \delta_{in}\delta_{jm}A_jB_mC_n = \delta_{im}B_mA_jC_j - \delta_{in}C_nA_jB_j = B_iA_jC_j - C_iA_jB_j$$

$$= B_i(\vec{A}.\vec{C}) - C_i(\vec{A}.\vec{B})$$

Similarly $[\vec{A}\times(\vec{B}\times\vec{C})]_j = B_j(\vec{A}.\vec{C}) - C_j(\vec{A}.\vec{B})$ and $[\vec{A}\times(\vec{B}\times\vec{C})]_k = B_k(\vec{A}.\vec{C}) - C_k(\vec{A}.\vec{B})$

Hence $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

Show that scalar product of two vector is a scalar (rotational invariance of dot product)

$$(\vec{A}.\vec{B})' = \sum_{i=1}^{3} A'_i B'_i = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\alpha_{ij}A_j\right)\left(\sum_{k=1}^{3}\alpha_{ik}B_k\right) = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}(\alpha_{ij}\alpha_{ik})A_jB_k$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{jk}A_jB_k = \sum_{j=1}^{3}A_jB_j = \vec{A}.\vec{B}$$

Show that vector product of two vector is a vector (rotational invariance of cross product)

$$(\vec{A}\times\vec{B})'_i = \xi'_{ijk}A'_jB'_k = (\boldsymbol{\alpha_{il}\alpha_{jm}\alpha_{kn}}\xi_{lmn})(\alpha_{jp}A_p)(\alpha_{kq}B_q) = \boldsymbol{\alpha_{il}}(\boldsymbol{\alpha_{jm}}\alpha_{jp})(\boldsymbol{\alpha_{kn}}\alpha_{kq})\xi_{lmn}A_pB_q$$

$$= \boldsymbol{\alpha_{il}}(\delta_{mp})(\delta_{nq})\xi_{lmn}A_pB_q = \boldsymbol{\alpha_{il}}\xi_{lmn}A_mB_n = \boldsymbol{\alpha_{il}}(\vec{A}\times\vec{B})_l$$

DYAD PRODUCT: A dyad product of introduced by Gibbs, is simple a pair of vectors written in definite order $\vec{A}\,\vec{B}$ without any dot(.) or cross ($\times$) in between them. $\vec{A}$ is called antecedent and $\vec{B}$ is called consequence. Dyad $\vec{A}\,\vec{B}$ is defined only in terms of dot product of an arbitrary vector $\vec{C}$ in the following two way

$$\overrightarrow{A\,B}.\vec{C} = \vec{A}\,(\overrightarrow{B.C}) \qquad \vec{C}\ AS\ POST-FACTOR$$

$$\vec{C}.\vec{A}\,\vec{B} = (\overrightarrow{C.A})\,\vec{B} \qquad \vec{C}\ AS\ PRE-FACTOR$$

Dyad product is an example of linear vector operator

Scalar and vector field:

A scalar field is defined as a region of space, with each point of which there is associated a scalar point function. As example- the temperature of points in the atmosphere at a given moment.

A vector field is defined as a region of space, with each point of which there is associated a vector point function.

As example- the wind velocity of points in the atmosphere at a given moment.

We denote scalar field by ϕ and vector field by A.

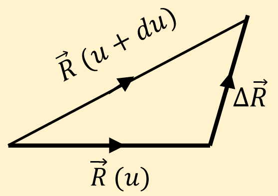
The vector differential operator: **Del** or **Nabla** is an operator used in mathematics, in particular in vector calculus as a vector differential operator usually represented by the $\vec{\nabla}$. When applied to a function defined on a 1D domain, it denotes its standard derivative as defined in calculus. When applied to a field (a function is defined in multi dimension domain), Del may denote the gradient (locally steepest slow) of a scalar field (or sometimes of a vector field as in the Navier-Stoke equation), the divergence of a vector field, or the curl (rotation) of a vector field depending on way it is applied.

Strictly speaking, Del is not a specific operator but rather a convenient mathematical notation for those three operators that makes many equations easier to write and remember. The del symbol can be interpreted as a vector of partial derivatives of operators, and its three possible meanings gradient, divergence and curl can be formally viewed as the product of scalars, dot product and cross product respectively of the del 'operator' with the field. This formal product do not necessarily commute with other operators or products.

In the Cartesian 3D coordinate system with coordinate (x, y, z) and standard basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ Del is defined as $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Vector Differentiation:

Ordinary Derivatives of vectors:



Let $R(u)$ be a vector depending on a single scalar variable u.

Then $\frac{\Delta\vec{R}}{\Delta u} = \frac{\vec{R}(u+\Delta u)-\vec{R}(\Delta u)}{\Delta u}$

where Δu denotes small increment of u.

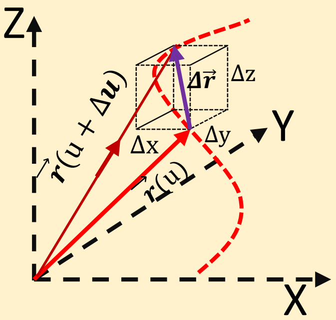
The ordinary derivative of the vector $R(u)$ with respect to the scalar u is given by,

$\frac{d\vec{R}}{du} = \lim_{\Delta u\to 0}\frac{\Delta\vec{R}}{\Delta u} = \lim_{\Delta u\to 0}\frac{\vec{R}(u+\Delta u)-\vec{R}(\Delta u)}{\Delta u}$, if the limit exists. Like this manner higher order derivatives are described.

Space Curves: If in particular $\vec{R}(u)$ is the position vector $\vec{r}(u)$ joining the origin O of a coordinate system and any point (x, y, z) then, $\vec{r}(u) = x(u)\hat{\imath} + y(u)\hat{\jmath} + z(u)\hat{k}$.

As u changes, the terminal point of $\vec{r}$ describes a space curve having parametric equation

$x = x(u), y = y(u), z = z(u)$



Then $\frac{\Delta\vec{r}}{\Delta u} = \frac{\vec{r}(u+\Delta u)-\vec{r}(u)}{\Delta u}$ is a vector in the direction of $\Delta\vec{r}$.

If $\lim_{\Delta u\to 0}\frac{\Delta\vec{r}}{\Delta u} = \frac{d\vec{r}}{du}$ exists,

the limit will be a vector in the direction of the tangent to the space curve at (x, y, z) and is given by, $\frac{d\vec{r}}{du} = \frac{dx}{du}\hat{\imath} + \frac{dy}{du}\hat{\jmath} + \frac{dz}{du}\hat{k}$

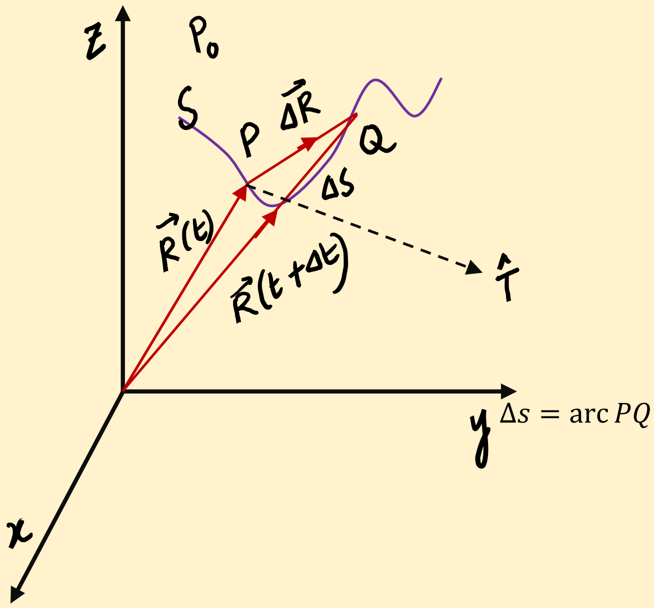
If u is the limit t, $\frac{d\vec{r}}{dt}$ represents the velocity v with which terminal point of $\vec{r}$ describes the curve.

Similarly, $\frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ represents its acceleration $\vec{a}$ along the curve.

(1)TANGENT Let a particle moves along the curve P_0PQ in space and P and Q be the positions of it at time t and $t + \Delta t$ respectively. Let $\vec{R}(t)$ and $\vec{R}(t+\Delta t)$ be the position vectors of P and Q with respect to the origin O of the Cartesian co-ordinate system X, Y, Z. Then, $\Delta\vec{R} = \vec{R}(t+\Delta t) - \vec{R}(t)$

$\frac{\Delta\vec{R}}{\Delta t}$ is directed along the chord PQ. In the limit when $\Delta t \to 0$, $\frac{\Delta\vec{R}}{\Delta t}$ becomes the tangent to the curve at P, if it exists.

Thus, $\lim_{\Delta t\to 0}\frac{\Delta\vec{R}}{\Delta t} = \frac{d\vec{R}}{dt}$ is a tangent vector to the space curve $\vec{R} = \vec{f}(t)$. If P_0 be the initial point corresponding to $t = t_0$ and s be the length of the arc P_0P,



Then $\frac{\Delta s}{\Delta t} = \frac{\Delta s}{|\Delta\vec{R}|}\cdot\frac{|\Delta\vec{R}|}{\Delta t} = \frac{\text{arcPQ}}{\text{chord } PQ}\left|\frac{\Delta\vec{R}}{\Delta t}\right|$,

Where

But, when $\Delta t \to 0$, Q apprонches the point P along the curve QP, then $\frac{\text{arc } PQ}{\text{chord } PQ} = 1$,

$\therefore \frac{ds}{dt} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0}\left|\frac{\Delta\vec{R}}{\Delta t}\right| = \left|\frac{d\vec{R}}{dt}\right|$

No: if $\frac{d\vec{R}}{dt}$ is is continuous, then P_0P is given by $s = \int_{t=t_0}^{t}\left|\frac{d\vec{R}}{dt}\right|dt = \int_{t=t_0}^{t}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$

Than the arc length s as the parameter, i.e., $s = t$, theh $\frac{ds}{dt} = 1$ and $\frac{d\vec{R}}{dt} = \frac{d\vec{R}}{ds}$. Then the magnitude of the tangent vector ($\vec{T}$) is $\left|\frac{d\vec{R}}{ds}\right| = 1$. Thus, the unit tangent vector is $\boldsymbol{\hat{T} = \frac{d\vec{R}}{ds}}$

(2) PRINCIPAL NORMAL

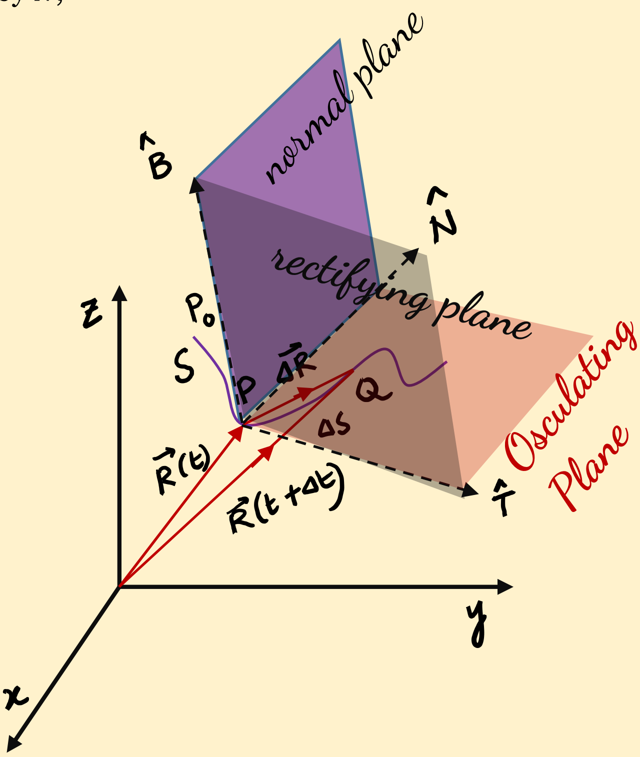
As $\hat{T}$ is a unit vector, $\therefore \hat{T}.\frac{d\hat{T}}{ds} = 0$ That means $\frac{d\hat{T}}{ds}$ perpendicular to $\hat{T}$ or, $\frac{d\hat{T}}{ds} = 0$ then $\hat{T}$ is a constant vector with respect to s, which means $\hat{T}$ has constant direction and hence the curve is a straight line.

If we denote the direction of $\frac{d\hat{T}}{ds}$ or, unit vector along $\frac{d\hat{T}}{ds}$ by $\hat{N}$,

then $\boxed{\frac{d\hat{T}}{ds} = \kappa\hat{N}, \ldots\ldots\ldots..(\text{i})}$ where $|\hat{N}| = 1$.

where the scalar κ is called the curvature.

The unit vector $\hat{N}$ is called the principal normal to the space curve at P. The plane containing $\hat{T}$ and $\hat{N}$ is called the Osculating plane of the curve.



(3) BINORMAL

The third unit vector given by $\hat{B} = \hat{T} \times \hat{N}$, is called the binormal at P. $\hat{B}$ is perpendicular to $\hat{T}$ and $\hat{N}$ and therefore, normal to the osculating plane at P of the space curve. Therefore, at each point P of the space curve there are three mutually perpendicular vectors $\hat{T}, \hat{N}$ and $\hat{B}$. $\hat{T}, \hat{N}$ and $\hat{B}$ form a right-handed set of unit vectors.

Hence, $\hat{T} = \hat{N} \times \hat{B}$, $\hat{N} = \hat{B} \times \hat{T}$ & $\hat{B} = \hat{T} \times \hat{N}$ This $\hat{T}, \hat{N}, \hat{B}$ constitute a moving tetrahedral which determines the following three mental planes at each point of the space curve $\vec{R}(t)$

(a) The osculating plane containing $\hat{T}$ and $\hat{N}$.

(b) The normal plane containing $\hat{N}$ and $\hat{B}$.

(c) The rectifying plane containing $\hat{B}$ and $\hat{T}$.

(4) CURVATURE

The curvature κ is the arc rate of turning of the tangent, given by $\frac{d\hat{T}}{ds}$, which is in the direction of the principal normal $\hat{N}$.

$$\boxed{\therefore \frac{d\hat{T}}{ds} = \kappa\hat{N}. \ldots\ldots\ldots..(\text{ii})}$$

where $\rho = \frac{1}{\kappa}$ is called the radius of curvature of the space curve.

(5) TORSION

We have $\boldsymbol{\frac{d\hat{B}}{ds} \cdot \hat{B} = 0,}$ since $\hat{B}$ is a unit vector.

Again, $\hat{B} \cdot \hat{T} = 0$ then, $\frac{d\hat{B}}{ds} \cdot \hat{T} + \hat{B} \cdot \frac{d\hat{T}}{ds} = 0$ or, $\frac{d\hat{B}}{ds} \cdot \hat{T} + \hat{B} \cdot \kappa \widehat{\mathrm{N}} = 0$

Hence $\boldsymbol{\frac{d\hat{B}}{ds} \cdot \hat{T} = 0}$ as $\hat{B} \cdot \widehat{\mathrm{N}} = 0$

hence $\frac{d\hat{B}}{ds}$ is perpendicular to both $\boldsymbol{\hat{T}}$ **and** $\boldsymbol{\hat{B}}$ **and parallel to** $\widehat{\mathrm{N}}$ the direction of $\frac{d\hat{B}}{ds}$ is in the negative direction of $\widehat{N}$.

$\frac{d\hat{B}}{ds} = -\tau\hat{N}$ where τ is called torsion . The quantity, $\sigma = \frac{1}{\tau}$ is called the radius of the torsion.

Now, since $\widehat{N} = \hat{B} \times \hat{T}$

$\therefore \frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\bar{T}}{ds} = -\tau\widehat{N} \times \hat{T} + \hat{B} \times \kappa\hat{N} = -\tau(-\hat{B}) + \kappa(-\hat{T})$

as $\hat{N} \times \hat{T} = -\hat{B}$ and $\hat{B} \times \widehat{N} = -\hat{T}$]

then $\boxed{\frac{d\widehat{N}}{ds} = \tau\widehat{B} - \kappa\hat{T} \text{ ………}}$(iii)

Equations (i), (ii) and (iii) are the ***FRENET-SERRET FORMULAE OF DIFFERENTIAL GEOMETRY***.

Continuity and Differentiability:

A scalar function $\phi(u)$ is called continuous at u if $|\phi(u+\Delta u)-\phi(\Delta u)|<\epsilon$ whenever $|\Delta u|<\delta$

A vector function $\vec{R}(u)$ is called continuous at u if $|\vec{r}(u+\Delta u)-\vec{r}(\Delta u)|<\epsilon$ whenever $|\Delta u|<\delta$

A scalar or vector function u is called differentiable of order n if its n^{th} derivative exists. A function which is differentiable is necessarily continuous but the converse is not true.

Differentiation Formula:

1. $\frac{d}{du}(\vec{A}+\vec{B})=\frac{d\vec{A}}{du}+\frac{d\vec{B}}{du}$ 2. $\frac{d}{du}(\vec{A}\cdot\vec{B})=\frac{d\vec{A}}{du}\cdot\vec{B}+\vec{A}\cdot\frac{d\vec{B}}{du}$ 3. $\frac{d}{du}(\vec{A}\times\vec{B})=\vec{A}\times\frac{d\vec{B}}{du}+\frac{d\vec{A}}{du}\times\vec{B}$ 4. $\frac{d}{du}(\phi\vec{A})=\phi\frac{d\vec{A}}{du}+\frac{d\phi}{du}\vec{A}$	5. $\frac{d}{du}(\vec{A}\cdot\vec{B}\times\vec{C})=\vec{A}\cdot\vec{B}\times\frac{d\vec{c}}{du}+\vec{A}\cdot\frac{d\vec{B}}{du}\times\vec{C}+\frac{d\vec{A}}{du}\cdot\vec{B}\times\vec{C}$ 6. $\frac{d}{du}[\vec{A}\times(\vec{B}\times\vec{C})]$ $=\vec{A}\times\vec{B}\times\frac{d\vec{c}}{du}+\vec{A}\times\left(\frac{d\vec{B}}{du}\times\vec{C}\right)+\frac{d\vec{A}}{du}\times(\vec{B}\times\vec{C})$

Partial Derivatives of vectors: If $\vec{A}$ is a vector depending on more than one scalar variable, say (x,y,z) for example, than we write $\vec{A}=\vec{A}(x,y,z)$. The partial derivative of $\vec{A}$ w.r.t to x is defined as, $\frac{\partial\vec{A}}{\partial x}=\lim_{\Delta x\to 0}\frac{\vec{A}(x+\Delta x,y,z)-\vec{A}(x,y,z)}{\Delta x}$, if the limit exist.

Similarly $\frac{\partial\vec{A}}{\partial y}=\lim_{\Delta y\to 0}\frac{\vec{A}(x,y+\Delta y,z)-\vec{A}(x,y,z)}{\Delta y}$ and $\frac{\partial\vec{A}}{\partial y}=\lim_{\Delta z\to 0}\frac{\vec{A}(x,y,z+\Delta z)-\vec{A}(x,y,z)}{\Delta z}$

Note that, $\phi(x,y)$ is called continuous at (x,y) if $|\phi(x+\Delta x,y+\Delta y)-\phi(x,y)|<\epsilon$ whenever $|\Delta x|<\delta$, $|\Delta y|<\delta$

For functions of two or more variable to mean that the function has continuous first partial derivatives.

Rules for partial differentiation of vectors are similar to those used in elementary calculus for scalar for scalar functions.

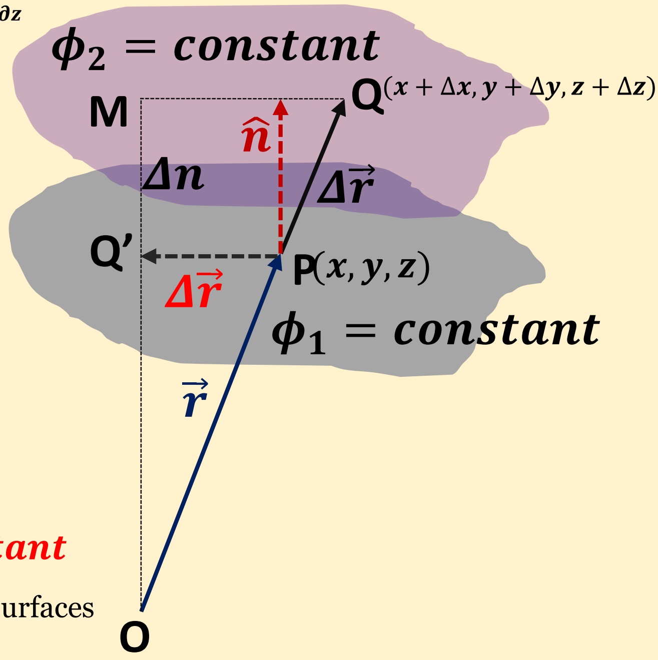
Higher derivatives can be defined as, $\frac{\partial^2\vec{A}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\right),\frac{\partial^2\vec{A}}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\right)$

Differential of a vector: If $\vec{A}$ is a vector $d\vec{A}$ is a differential of $\vec{A}$. If $\vec{A}=\vec{A}\,(x,y,z)$

then $d\vec{A}=\frac{\partial A_x}{\partial x}dx+\frac{\partial A_y}{\partial y}dy+\frac{\partial A_z}{\partial z}dz.$

Now $d(\vec{A}\cdot\vec{B})$= homework, $d(\vec{A}\times\vec{B})$= homework.

Gradient:



Gradient of $\boldsymbol{\phi} \equiv$ grad $\boldsymbol{\phi} \equiv \vec{\nabla}\boldsymbol{\phi}$ **where** $\vec{\nabla} \equiv \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Here $\boldsymbol{\phi} = \boldsymbol{\phi}(x,y,z)$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Then $\Delta\boldsymbol{\phi} = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z = \vec{\nabla}\boldsymbol{\phi}.\Delta\vec{r}$

If point Q lies on the level surface at Q' Then $\Delta\phi = 0$,

hence $\vec{\nabla}\phi \cdot \Delta\vec{r} = 0$

Hence $\vec{\nabla}\phi$ is perpendicular to the level surface.

Then $\vec{\nabla}\boldsymbol{\phi} = |\vec{\nabla}\boldsymbol{\phi}|\hat{\boldsymbol{n}}$

Where $\hat{n}$ is the unit normal to the surface $\boldsymbol{\phi = constant}$

Let Δn be the perpendicular distance between the level surfaces

In figure and $\hat{n}$ is the unit normal to the surface S at P

Now $\frac{d\Phi}{dn} = \lim_{\Delta n\to 0}\frac{\Delta\Phi}{\Delta n} = \lim_{\Delta n\to 0}\frac{\nabla\phi\cdot\Delta\vec{r}}{\Delta n}$

$= |\vec{\nabla}\phi|\lim_{\Delta n\to 0}\frac{\hat{n}\cdot\Delta\vec{r}}{\Delta n} = |\vec{\nabla}\phi|\lim_{\Delta n\to 0}\frac{\Delta n}{\Delta n} = |\vec{\nabla}\phi|$

Hence $\vec{\nabla}\phi = \frac{d\phi}{dn}\hat{n}$

The gradient of a scalar function $\boldsymbol{\phi}$ is a vector whose magnitude is equal to the maximum rate of change of $\boldsymbol{\phi}$ w.rt the space co-ordinate Variables and whose direction is along that change (normal to the level surface S).

Directional derivatives and normal derivative

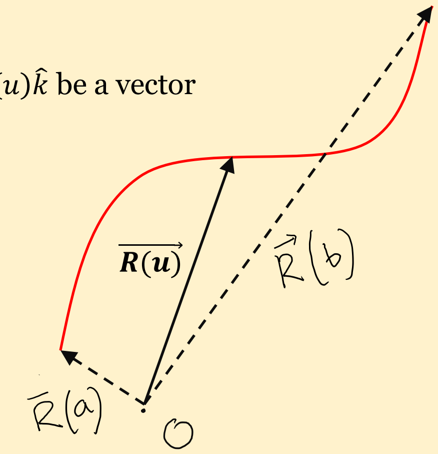
Normal derivative: if $\phi(x,y,z) = C$ represents a family of surfaces for different values of constant C. On differentiating ϕ we get $d\phi = 0$

But $\vec{\nabla}\phi.\overrightarrow{dr} = d\phi$ then in this case $\vec{\nabla}\phi.\overrightarrow{dr} = 0$

the scalar product of two vectors $\vec{\nabla}\phi$ & $\overrightarrow{dr}$are being zero. Hence $\vec{\nabla}\phi$ & $\overrightarrow{dr}$ perpendicular to each other. $\overrightarrow{dr}$ is in the direction of tangent of the given surface. Thus $\vec{\nabla}\phi$ is a vector normal to the surface.

Directional derivative: The component of $\vec{\nabla}\phi$ in the direction of the vector $\vec{d}$ is equal to$\vec{\nabla}\phi.\hat{d}$ and is called directional derivative of in the direction of $\vec{d}$

Vector Integration:



Ordinary integrals of vector: Let $\vec{R}(u) = R_1(u)\hat{\imath} + R_2(u)\hat{\jmath} + R_3(u)\hat{k}$ be a vector depending supposed continuous in a specified interval.

Then, $\int \vec{R}(u)du = \hat{\imath}\int R_1(u)du + \hat{\jmath}\int R_2(u)du + \hat{k}\int R_3(u)du$ is called an indefinite integral of $\vec{R}(u)$.

If there exists a vector $\vec{S}(u)$ such that $\vec{R}(u) = \frac{d}{du}\left(\vec{S}(u)\right)$,

then $\int \vec{R}(u)du = \int \frac{d}{du}\left(\vec{S}(u)\right) = \vec{S}(u) + \vec{C}$

Where C is an arbitrary const. vector independent of u.

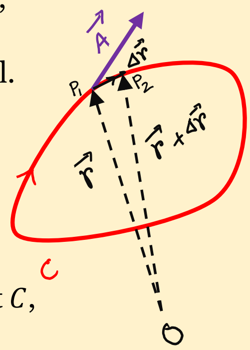
The definite integral between limits $u = a$ and $u = b$ can in such case be written,

$$\int_a^b \vec{R}(u)du = \int_a^b \frac{d}{du}\left(\vec{S}(u)\right)du = \vec{S}(u) + [\vec{C}]_a^b = \vec{S}(b) - \vec{S}(a)$$

This integral can be also defined as a limit of a sum in a manner analogous to that of elementary integral calculus.

Line integrals of vector: Let $\vec{r}(u) = x(u)\hat{\imath} + y(u)\hat{\jmath} + z(u)\hat{k}$ where $\vec{r}(u)$the position vector of is (x, y, z) define a curve C joining points P_1 to P_2 where $u = u_1$ *and* $u = u_2$ respectively. We assume that C is composed of a finite number of curves for each of which $\vec{r}(u)$ continues derivative.

Let $\vec{A}(x, y, z) = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\vec{A}$ along C from P_1 to P_2 written as,



$\int_{P_2}^{P_1} \vec{A}\cdot dr = \int_C \vec{A}\cdot dr = \int A_1dx + \int A_2dy + \int A_3dz$ is an example of line integral.

If C is a closed curve the integral around C is often denoted by

$$\oint \vec{A}\cdot d\vec{r} = \int A_1dx + \int A_2dy + \int A_3dz$$

In aerodynamics and fluid mechanics this integral is called circular of $\vec{A}$ about C, where $\vec{A}$ represent the velocity of fluid.

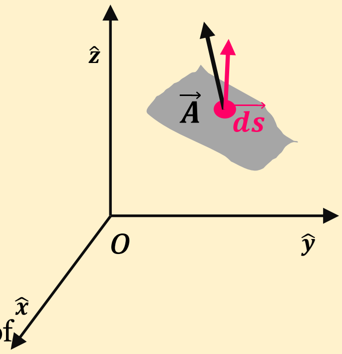
In general, any integral which is to be evaluated along a curve is called line integral.

Such integral can be defined in term of limit of sums as are the integrals of elementary calculus.

Surface integrals of vector: Let S be two sided surface, such as shown in figure below. Let one side of S to be considered arbitrarily as positive side [if S is a closed surface] this is taken as outer side. A unit normal $\hat{n}$ to any point of the positive side of S is called a positive outward drawn unit normal. Associated with the differential of surface area dS over a vector $d\vec{S}$ whose magnitude is dS and whose direction is that of $\hat{n}$ then $d\vec{S} = dS\hat{n}$

The outward flux of the continuous vector function $\vec{A}$ through dS is $dN = \vec{A}\cdot\overrightarrow{dS}$.

The outward flux through the whole surface S is$N = \iint \vec{A}\cdot d\vec{S}$



The integral $\iint \vec{A}\cdot d\vec{S} = \iint \vec{A}\cdot\hat{n}d\vec{S}$ is an example of surface integral called flux of $\vec{A}$ over S.

Other surface integral are: $\iint \phi d\vec{S}$, $\iint \vec{A}\times d\vec{S}$

where $\boldsymbol{\phi}$ is a scalar function such integral can be defined in terms of limit of sums are in the elementary calculus. The notation $\oiint$ is sometimes use to indicate integration over the closed surface S.

$\iint \vec{A}\cdot d\vec{S}$ as the limit of sum over.

We consider the outward surface S of a body in a vector field $\vec{A}$. Let ΔS_p be an element of surface S surrounding the point P_p where the vector field $\overrightarrow{A_p}$ is and the outward unit normal is $\widehat{n_p}$

The normal flux of $\overrightarrow{A_p}$ is $\sum_{p=1}^{M}\overrightarrow{A_p}\,.\,\Delta S_p\,\widehat{n_p}$ where $p = 1,2,3 \ldots\ldots M$

When $M\to\infty$, the largest $\Delta S_p \to 0$, if the limit exists then,

$\iint_S \vec{A}\cdot\hat{n}dS$ is called the thet integral of the normal component of $\vec{A}$ oyer S.

To Show that If the surface S has projection R on the xy-plane, then $\iint_S \vec{A}\cdot\hat{n}dS = \iint_R \vec{A}\cdot\hat{n}\frac{dxdy}{|\hat{n}_p\cdot k|}$

Proof: From Fig. the projection of ΔS_p on the xy-plane is $\left|\left(\widehat{n_p}\Delta S_p\right)\cdot\vec{k}\right| = \left|\hat{n}_p\cdot\hat{k}\right|\Delta S_p$.

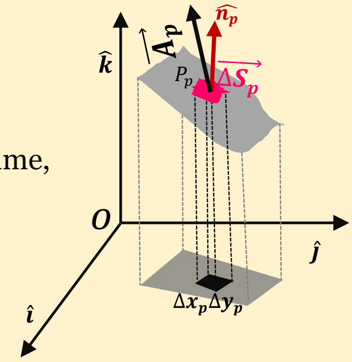
But, $(\hat{n}_p\cdot\hat{k})\Delta S_p = \Delta x_p\Delta y_p \;\therefore\; \Delta S_p = \frac{\Delta x_p\Delta y_p}{|\hat{n}_p\cdot\hat{k}|}$

Using this value of ΔS_p we get $\sum_{p=1}^{M}\vec{A}_p\cdot\hat{n}_p\frac{\Delta x_p\Delta y_p}{|\hat{n}_p\cdot\hat{k}|}$

In the limit, when $M\to\infty$ largest Δx_p and Δy_p tend to zero at the same time,

we have $\sum_{j=1}^{M}\vec{A}_p\cdot\hat{n}_p\frac{\Delta x_p\Delta y_p}{|\hat{n}_p\cdot\hat{k}|} = \iint_R \vec{A}\cdot\hat{n}\frac{dxdy}{|\hat{n}_p-\hat{k}|}$

$$\iint_S \vec{A}\cdot\hat{n}dS = \iint_R \vec{A}\cdot\hat{n}\frac{dxdy}{|n_p\cdot k|}.$$



Volume integrals of vector: Consider a closed surface in space inclosing a volume V. Then $\iiint_V \vec{A}\, dV$ and $\iiint_V \phi\, dV$ are examples of volume integrals.

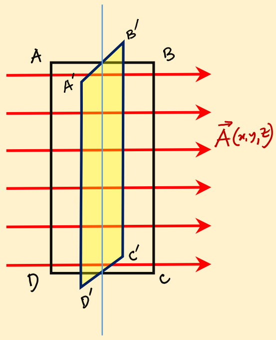
$\iiint_V \vec{A}\, dV = \hat{\imath} \iiint A_1\, dV + \hat{\jmath} \iiint A_2\, dV + \hat{k} \iiint A_3\, dV$ where $\vec{A} = \hat{\imath}\, A_1 + \hat{\jmath} A_2 + \hat{k} A_3$

Curl of a Vector field and its significance: We have seen above that for a vector field derivative from a scalar field (function), the line integral over a closed path of vector is zero. Such a vector field called **Lamellar vector field.**

But for other types of vector field line integral has some values and called **non-lamellar vector field.**

The amount of maximum line integral at any point in a vector field around a close curve when expressed for unit area is called the curl or rotation of the vector field at the point and is associated with the vector sense of positive normal drawn on the small exploring area in the position of the maximum line integral.

By definition, $\vec{\nabla} \times \vec{A} = \lim\limits_{\Delta S \to 0} \frac{\oint \vec{A} \cdot d\vec{r}}{\Delta S}$



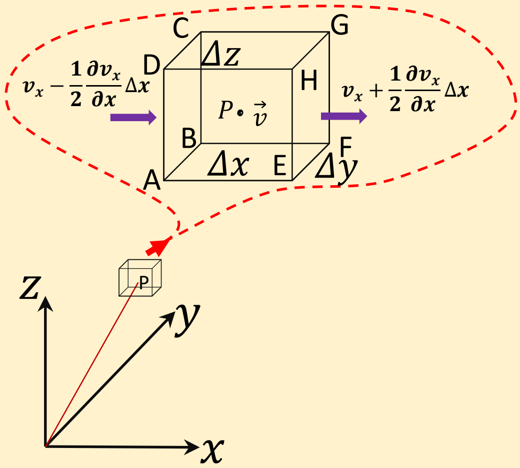
The adjoining figure shows some vectors lines. A small rectangular area is shown in two position (i) $ABCD$ $(ii) A'B'C'D'$. The area $A'B'C'D'$ is perpendicular to the field, so the contribution to the line integral is zero. In the other case ABCD is parallel to the field and line integral along AB and CD has finite values but line integral of AD and CB are zero. Thus there is a certain orientation of the area for which the line integral is greatest. The greatest line integral for unit area is called curl of a vector field. And is denoted by rot$\vec{A}$ or $\vec{\nabla} \times \vec{A}$. This is a vector quantity along the normal to the exploring area.

Note that when $\vec{\nabla} \times \vec{A} = \vec{0}$ then is caller ***irrotational,*** if $\vec{A}$ represent any force field then the force field $\vec{A}$ which satisfying $\vec{\nabla} \times \vec{A} = \vec{0}$ is called ***conservative force field.***

Divergence of a Vector field and its significance: The divergence of a vector field is defined as the net amount of the flux of the vector field diverging or converging per unit volume. Let consider a fluid moves and its velocity at any point $P(x, y, z)$ is $\vec{V} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k} = \vec{V}(x, y, z)$.

We consider parallelepiped having center P and edges parallel to coordinate axes with magnitude $\Delta x, \Delta y, \Delta z$ respectively as shown in figure.

Let $\frac{\partial v_x}{\partial x}, \frac{\partial v_y}{\partial y}, \frac{\partial v_z}{\partial z}$ be the velocity gradient along x, y, z axes. From figure X-component of velocity at P is v_x,



Hence X-component of velocity at the face $ABCD$

$= v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x$.....(i)

Similarly X-component of velocity at the face $EFGH$

$= v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x$.....(ii)

∴ Volume of the fluid crossing ABCD per unit time

$= \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y \Delta z$ and

Volume of the fluid crossing ABCD per unit time

$= \left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y \Delta z$

Loss in volume per unit time along X-axis,

$\left\{\left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right) - \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\right\}\Delta y \Delta z \;= \frac{\partial v_x}{\partial x}\Delta x \Delta y \Delta z$

Similarly along Y and Z axes the quantities will be $\frac{\partial v_y}{\partial y}\Delta x \Delta y \Delta z$ and $\frac{\partial v_z}{\partial z}\Delta x \Delta y \Delta z$

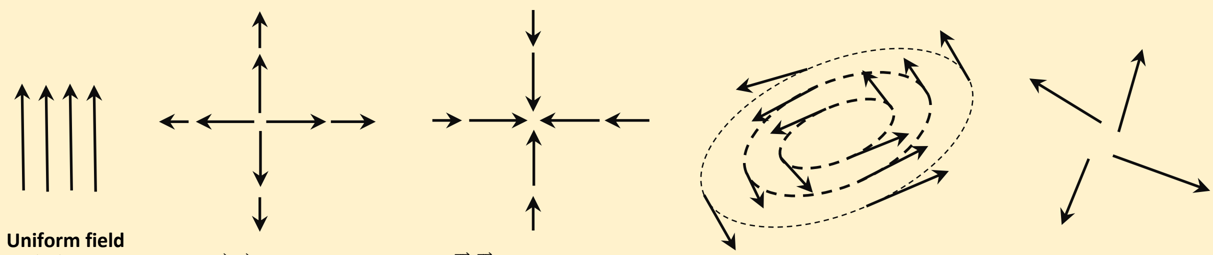
Hence total loss per unit volume$= \frac{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = \vec{\nabla}\cdot\vec{V}$

Note that, if ρ be the density of the compressible fluid moving with velocity $\vec{V}$, then obviously

$\vec{\nabla}\cdot\vec{V} + \frac{\partial \rho}{\partial t} = 0$ which is continuity equation. This equation states that the net flow out of volume results a decrease in density inside volume.

By definition, $\vec{\nabla}\cdot\vec{A} = \lim_{\Delta V \to 0}\frac{\oint \vec{A}\cdot d\vec{r}}{\Delta V}$.   If $\vec{\nabla}\cdot\vec{A} = 0$ the <u>***$\vec{A}$ is said to be Solenoidal,***</u> if $\vec{\nabla}\cdot\vec{A} > 0$ the P is a source point.

If $\vec{\nabla}\cdot\vec{A} < 0$ the P sink point.



Uniform field $\vec{\nabla}.\vec{A} = 0$, $\vec{\nabla}\times\vec{A} = 0$ | $\vec{\nabla}.\vec{A} \neq 0\ (+ve)$, $\vec{\nabla}\times\vec{A} = 0$ | $\vec{\nabla}.\vec{A} \neq 0\ (-ve)$, $\vec{\nabla}\times\vec{A} = 0$ | $\vec{\nabla}.\vec{A} = 0$, $\vec{\nabla}\times\vec{A} \neq 0$ | $\vec{\nabla}.\vec{A} \neq 0\ (+ve)$, $\vec{\nabla}\times\vec{A} \neq 0$

PROOF OF VECTOR IDENTITIES:

1. $\vec{\nabla}(f+g)=\vec{\nabla}(f)+\vec{\nabla}(g)$

➢ $\vec{\nabla}(f+g)=\hat{x}_i\frac{\partial}{\partial x_i}(f+g)=\hat{x}_i\frac{\partial}{\partial x_i}(f)+\hat{x}_i\frac{\partial}{\partial x_i}(g)=\vec{\nabla}(f)+\vec{\nabla}(g)$

2. $\vec{\nabla}(fg)=f\vec{\nabla}(g)+g\vec{\nabla}(f)$

➢ $\vec{\nabla}(fg)=\hat{x}_i\frac{\partial}{\partial x_i}(fg)=\hat{x}_i g\frac{\partial}{\partial x_i}(f)+\hat{x}_i f\frac{\partial}{\partial x_i}(g)=f\vec{\nabla}(g)+g\vec{\nabla}(f)$

3. $\vec{\nabla}\left(\frac{f}{g}\right)=\frac{g\vec{\nabla}(f)-f\vec{\nabla}(g)}{g^2}$

➢ $\vec{\nabla}\left(\frac{f}{g}\right)=\hat{x}_i\frac{\partial}{\partial x_i}\left(\frac{f}{g}\right)=\hat{x}_i\left\{\frac{g\frac{\partial}{\partial x_i}(f)-f\frac{\partial}{\partial x_i}(g)}{g^2}\right\}=\left\{\frac{g\hat{x}_i\frac{\partial}{\partial x_i}(f)-f\hat{x}_i\frac{\partial}{\partial x_i}(g)}{g^2}\right\}=\frac{g\vec{\nabla}(f)-f\vec{\nabla}(g)}{g^2}$

4. $\vec{\nabla}.(\vec{A}+\vec{B})=\vec{\nabla}.(\vec{A})+\vec{\nabla}.(\vec{B})$

➢ $\vec{\nabla}.(\vec{A}+\vec{B})=\frac{\partial}{\partial x_i}(\vec{A}+\vec{B})_i=\frac{\partial}{\partial x_i}\{(\vec{A})_i+(\vec{B})_i\}=\frac{\partial}{\partial x_i}(\vec{A})_i+\frac{\partial}{\partial x_i}(\vec{B})_i=\vec{\nabla}.(\vec{A})+\vec{\nabla}.(\vec{B})$

5. $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times(\vec{A})+\vec{\nabla}\times(\vec{B})$

➢ $[\vec{\nabla}\times(\vec{A}+\vec{B})]_i=\epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{A}+\vec{B})_k=\epsilon_{ijk}\frac{\partial}{\partial x_j}\left((\vec{A})_k+(\vec{B})_k\right)=\epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{A})_k+\epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{B})_k$

$=[\vec{\nabla}\times(\vec{A})]_i+[\vec{\nabla}\times(\vec{B})]_i$

Then $[\vec{\nabla}\times(\vec{A}+\vec{B})]_j=[\vec{\nabla}\times(\vec{A})]_j+[\vec{\nabla}\times(\vec{B})]_j$ and $[\vec{\nabla}\times(\vec{A}+\vec{B})]_k=[\vec{\nabla}\times(\vec{A})]_k+[\vec{\nabla}\times(\vec{B})]_k$

Hence $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times(\vec{A})+\vec{\nabla}\times(\vec{B})$

6. $\vec{\nabla}.(k\vec{A})=k\vec{\nabla}.(\vec{A})+\vec{A}.\vec{\nabla}(k)$

➢ $\vec{\nabla}.(k\vec{A})=\frac{\partial}{\partial x_i}(k\vec{A})_i=\frac{\partial}{\partial x_i}k(\vec{A})_i=k\frac{\partial}{\partial x_i}(\vec{A})_i+(\vec{A})_i\frac{\partial}{\partial x_i}k=k\vec{\nabla}.(\vec{A})+\vec{A}.\vec{\nabla}(k)$

7. $\vec{\nabla}\times(k\vec{A})=k\vec{\nabla}\times(\vec{A})+\vec{A}\times\vec{\nabla}(k)$

➢ $[\vec{\nabla}\times(k\vec{A})]_i=\epsilon_{ijk}\frac{\partial}{\partial x_j}(k\vec{A})_k=k\epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{A})_k+\epsilon_{ijk}(\vec{A})_k\frac{\partial}{\partial x_j}k=k[\vec{\nabla}\times(\vec{A})]_i+[\vec{A}\times\vec{\nabla}(k)]_i$

8. $\vec{\nabla}(\vec{A}.\vec{B})=(\vec{B}.\vec{\nabla})\vec{A}+(\vec{A}.\vec{\nabla})\vec{B}+\vec{B}\times(\vec{\nabla}\times\vec{A})+\vec{A}\times(\vec{\nabla}\times\vec{B})$

➢ $(\vec{\nabla}(\vec{A}\cdot\vec{B}))_i=\frac{\partial}{\partial x_i}(\vec{A}\cdot\vec{B})=\frac{\partial}{\partial x_i}(A_j\cdot B_j)=\frac{\partial A_j}{\partial x_i}B_j+A_j\cdot\frac{\partial B_j}{\partial x_i}==(\nabla_i A_j)B_j+A_j(\vec{\nabla}_i B_j)$

Now $((\bar{B}\cdot\vec{\nabla})\bar{A})_i=B_j\cdot\frac{\partial}{\partial x_j}A_i=B_j(\nabla_j A_i)$

$((\bar{A}\cdot\vec{\nabla})\bar{B})_i=A_j(\nabla_j B_i)$

$(\bar{B}\times(\vec{\nabla}\times\bar{A})_i=\epsilon_{ijk}B_j(\vec{\nabla}\times\bar{A})_k=\epsilon_{ijk}\epsilon_{klm}B_j\nabla_l A_m=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})B_j\nabla_l A_m$

$=B_m\nabla_i A_m-B_j\nabla_j A_i$

Similarly $(\bar{A} \times (\bar{\nabla} \times \bar{B})_i = A_m \nabla_i B_m - A_j \nabla_j B_i$

Hence R.H.S= $B_j(\nabla_j A_i) + A_j(\nabla_j B_i) + B_m \nabla_i A_m - B_j \nabla_j A_i + A_m \nabla_i B_m - A_j \nabla_j B_i = B_m \nabla_i A_m + A_m \nabla_i B_m$

$= B_j\,(\nabla_i A_j) + A_j(\bar{\nabla}_i B_j)$ = L.H.S [replacing m to j]

9. $\vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})$

➢ $\vec{\nabla}.(\vec{A} \times \vec{B}) = \frac{\partial}{\partial x_i}(\vec{A} \times \vec{B})_i = \frac{\partial}{\partial x_i} \epsilon_{ijk} A_j B_k = \epsilon_{ijk} A_j \frac{\partial}{\partial x_i} B_k + \epsilon_{ijk} B_k \frac{\partial}{\partial x_i} A_j$

$= \; -A_j \epsilon_{jik} \frac{\partial}{\partial x_i} B_k + B_k \epsilon_{kij} \frac{\partial}{\partial x_i} A_j = -A_j(\vec{\nabla} \times \vec{B})_j + B_k(\vec{\nabla} \times \vec{A})_k$

$= -\vec{A}.(\vec{\nabla} \times \vec{B}) + \vec{B}.(\vec{\nabla} \times \vec{A}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})$

10. $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - \vec{B}\,(\vec{\nabla}.\vec{A}) - \;(\vec{A}.\vec{\nabla})\vec{B} + \vec{A}\,(\vec{\nabla}.\vec{B})$

➢ $(\nabla \times (A \times B))_i = \varepsilon_{ijk} \partial_j (A \times B)_k$

But $(A \times B)_k = \varepsilon_{klm} A_l B_m$

Thus,

$(\nabla \times (A \times B))_i = \varepsilon_{ijk} \partial_j (\varepsilon_{klm} A_l B_m)$

Recall:

$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Thus,

$(\nabla \times (A \times B))_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j(A_l B_m)$

$= \partial_j(A_i B_j) - \partial_j(A_j B_i)$

Expand each term:

$\partial_j(A_i B_j) = (\partial_j A_i) B_j + A_i(\partial_j B_j)$

$\partial_j(A_j B_i) = (\partial_j A_j) B_i + A_j(\partial_j B_i)$

Thus,

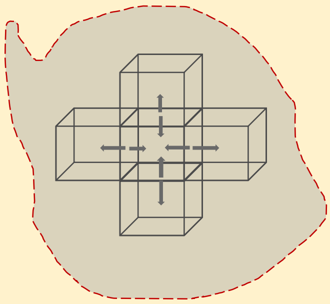
$(\nabla \times (A \times B))_i = (\partial_j A_i) B_j + A_i(\partial_j B_j) - (\partial_j A_j) B_i - A_j(\partial_j B_i)$

Thus,

$\nabla \times (A \times B) = (B \cdot \nabla) A - B(\nabla \cdot A) - (A \cdot \nabla) B + A(\nabla \cdot B)$

Gauss's Divergence theorem: Gauss's divergence theorem enables us to transform a volume integral into surface integral. If states that the volume integral of the divergence of a vector field $\vec{A}$ is taken over any volume V is equal to the surface integral of $\vec{A}$ taken over the closed surface surrounding the volume V.

i.e. $\iiint_V (\vec{\nabla}\cdot\vec{A})dV = \oiint \vec{A}\cdot d\vec{S}$



Imagine that volume V is sub divided into an arbitrary large number of tiny (differential) parallelepiped. For each parallelepiped

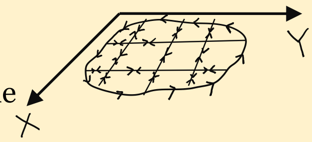
$\sum_{Six\ surface} \vec{A}\cdot d\vec{S} = \vec{\nabla}\cdot\vec{A}dV$

[The summation is over the six faces of the parallelepiped].

Summing over all parallelepiped, we find that, $\vec{A}\cdot d\vec{S}$ terms cancel (pairwise) for all interior faces; only the contribution of a Riemann integral as the limit of a sum, we take the limit as the number of parallelepiped approaches infinity ($\rightarrow\infty$) and the dimension of each approach zero ($\rightarrow 0$).

$\sum_{Exterior\ surface} \vec{A}\cdot d\vec{S} = \sum_{Volume}(\vec{\nabla}\cdot\vec{A})dV \quad\Rightarrow\quad \oiint \vec{A}\cdot d\vec{S} = \iiint(\vec{\nabla}\cdot\vec{A})dV$

Stoke's Theorem: Let us take a surface and sub-divided into a network or arbitrary small rectangles, we showed that circulation about such a differential rectangle (in the XY-plane) is $[(\vec{\nabla}\times\vec{A})]_z dxdy$ for one differential rectangle, then $\sum_{Foursides} \vec{A}\cdot d\vec{l} = (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$...(i)



If we sum over all the little rectangles as in the definition of Riemann integral. The surface contribution are added together. The line integral of all interior line segments cancel identically. Only the line integral around perimeter survives. Taking usual limit as the number of rectangles approaches infinity while $dx\rightarrow 0, dy\rightarrow 0$.

We have $\sum_{Exterior\ line\ segment} \vec{A}\cdot d\vec{l} = \sum_{Rectangles}(\vec{\nabla}\times\vec{A})\cdot d\vec{S} \quad\Rightarrow\quad \oiint \vec{A}\cdot d\vec{l} = \iiint(\vec{\nabla}\times\vec{A})\cdot d\vec{S}$

Statement:- This theorem states that the circulation of vector $\vec{A}$ around the contour can be replaced by surface integral 'pulled over' the contour. Mathematically, $\oint \vec{A}\cdot d\vec{l} = \iiint(\vec{\nabla}\times\vec{A})\cdot d\vec{S}$.

Green's Theorem: Using Gauss's divergence theorem we get another important theorem known as Green's theorem.

Suppose we write $\vec{A} = \psi\vec{\nabla}\phi$ where ψ and $\boldsymbol{\phi}$ are scalar functions. Then divergence gives,

$\oiint \vec{A}\cdot d\vec{S} = \iiint(\vec{\nabla}\cdot\vec{A})dV$ Now $\vec{\nabla}\cdot\vec{A} = \psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$

$\therefore \oiint \psi\vec{\nabla}\phi\cdot d\vec{S} = \iiint(\psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi)dV$.........(i)

Similarly if $\vec{A} = \phi\vec{\nabla}\psi$ then, $\oiint \phi\vec{\nabla}\psi\cdot d\vec{S} = \iiint(\phi\nabla^2\psi + \vec{\nabla}\psi\cdot\vec{\nabla}\phi)dV$.......(ii)

From (i) and (ii) we have, $\oiint(\psi\vec{\nabla}\phi - \phi\vec{\nabla}\psi)d\vec{S} = \iiint(\psi\nabla^2\phi - \phi\nabla^2\psi)dV$

This relation is referred to as second for of Green's theorem.

Green's Theorem in a Plane: If S is a closed region of the XY-plane bounded by a close curve C and if M and N are continuous function of x and y having continuous derivatives, then $\oint(Mdx + Ndy) = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$

Proof: Stokes theorem states that $\oint \vec{A}\cdot d\vec{r} = \iint_s(\vec{\nabla}\times\vec{A})\cdot\hat{n}dS$......(i)

If curve is plotted in XY-plane, then $\hat{n} = \hat{k}$ and $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$.

Let $A_n = M$ and $A_m = N$ then $\vec{\nabla}\times\vec{A} = \hat{\imath}\left(0 - \frac{\partial N}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial M}{\partial z} - 0\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$\therefore (\vec{\nabla}\times\vec{A})\cdot\hat{n} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

Again as $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ then $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$

Hence from equation (i), $\oint(Mdx + Ndy) = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Green's Theorem General Form: This theorem enables us to express an integral taken the space between them. If u, v, w are continuous function of (x, y, z) then, $\sum\iint(lu + mv + nw)\,dS = -\iiint\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dxdydz$

Hence $\sum$ denotes that the surface integrals are summed over any number of close surfaces l, m, n are the direction cosines of the normal drawn in every case from the element dS into space between the surfaces. The volume integral is taken throughout the space between surface

Multiple Integral:-

Double Integral: $I = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)dx\right]dy \quad or, \quad I = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)dy\right]dx$

Usually 1. When x_1 and x_2 are function of y and y_1 and y_2 are constant then $f(x,y)$ is first integrated w.r.t x, treating y constant, between the given limits and resulting expression is next integrated w.r.t y, between the limits.

2. When y_1 and y_2 are function of x and x_1 and x_2 are constant then $f(x,y)$ is first integrated w.r.t x, treating y constant, between the given limits and resulting expression is next integrated w.r.t y, between the limits.

3. When both the pairs of limits are constant, it is immaterial if the function is integrated first w.r.t x then y respectively or conversely.

Double integral in polar co-ordinate- $I = \int_{r_1}^{r_2}\left[\int_{\theta_1}^{\theta_2} f(r,\theta)d\theta\right]dr \; or, \quad I = \int_{\theta_1}^{\theta_2}\left[\int_{r_1}^{r_2} f(r,\theta)dr\right]d\theta$

Triple Integral: If x_1 and x_2 are constant, y_1 and y_2 are either constant or function of x, y. Then, $I = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2}\left[\int_{z_1}^{z_2} f(x,y,z)dzdydx\right]\right]$

Change of order of integration: In case of double integral with variable limit, the change of order of integration leads to change of limits of integration. The double integrals which seems to be complicated can be solved easily by splitting the region of integration and the given integral express as the sum of a number of double integrals.

To find the new limits of integration, we draw a rough sketch of the region of integration.

Change of variables:

1. Suppose $x = x(u,v)$ & $y = y(u,v)$ then, $\int_{R_{xy}} f(x,y)dxdy = \iint_{R_{uv}} f[x(u,v), y(u,v)]\,|J|\,dudv$

   where $|J| = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$
2. Suppose $x = x(u,v,w),\; y = y(u,v,w),\; z = z(u,v,w)$
   then, $\int_{R_{xyz}} f(x,y,z)dxdydz = \iint_{R_{uvw}} f[x(u,v,w), y(u,v,w), z(u,v,w)]\,|J|\,dudvdw$

   where $|J| = \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$

Special case:

(i) Cartesian t polar coordinate: Here $x = r\cos\theta,\; y = r\sin\theta$ then $\frac{\partial(x,y)}{\partial(r,\theta)} = r$

(ii) Rectangular to cylindrical coordinate: Here $x = \rho\cos\phi,\; y = \rho\sin\phi, z = z$ then $|J| = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = P$

(iii)Rectangular to spherical polar coordinate: Here $x = r\sin\theta\cos\phi,\; y = r\sin\theta\sin\phi, z = r\cos\theta$ then $|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Area enclosed by a 'plane curve':

1. Area $A = \int_{x_1}^{x_2}\int_{f_1(x)}^{f_2(x)} dxdy \quad or, \quad \int_{y_1}^{y_2}\int_{f_1(y)}^{f_2(y)} dxdy$ in Cartesian form.
2. Area $A = \iint rd\theta dr$ (with proper limit) in polar form.

Area of a 'Curved surface': Let $z = f(x, y)$ be the surface S and P be appoint on it. Let its projection on XY-plane be the region A, in which $dxdy$ is a elementary area then, $S = \iint \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dxdy$

Volume of Solids: -

1. **Volume as double integral:** Let $z = f(x, y)$ represents a surface $S'$and S be its orthogonal projection on XY-plane then volume of the solid cylinder with S as its base and bounded by a given surface, with generator parallel to z-axis, is
   $V = \iint zdxdy = \iint f(x, y)dxdy$ in Cartesian form
   $V = \iint rdrd\theta$ in polar form.
2. **Volume as triple integral:** $V = \iiint dxdydz \quad or, \iiint \rho d\rho d\phi dz \quad or, \iiint r^2 \sin\theta\, drd\theta d\phi$
3. **Volume of solids of revolution:** Let us consider an elementary area $\delta x\delta y$ on a plane area A. The revolution of $\delta x\delta y$ about X-axis will generate a ring of volume, $dV = \pi[(y + \delta y)^2 - y^2]\delta x \approx 2\pi y\delta x\delta y$.
   The revolution about X-axis will generate a volume $V_1 = \iint_A 2\pi x dxdy$
   Similarly, the revolution about Y-axis generate a volume $V_2 = \iint_A 2\pi y dxdy$
   In polar coordinate this volume is $V_3 = \iint_A 2\pi r \sin\theta\, rd\theta dr$ About X-axis
   $= \iint_A 2\pi r \cos\theta\, rd\theta dr$ About Y-axis