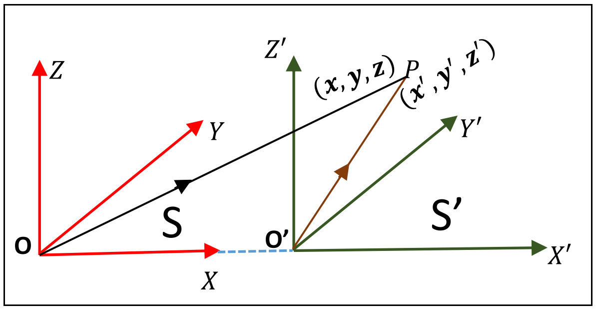
Galilean transformation.

The law of nature look exactly the same for all observers in the inertial frames, regardless of their state of relative velocity.



If the S' frame moving with a velocity v_0 along X-axis then the transformation equation becomes

$$x' = x - v_0 t; \quad y' = y; \quad z' = z; \quad t' = t$$

Differentiating both side with respect to t we get $\frac{dx'}{dt} = \frac{dx}{dt} - v_0, \frac{dy'}{dt} = \frac{dy}{dt}, \frac{dz'}{dt} = \frac{dz}{dt}$

Then transformation equation for velocity becomes

$$v'_x = v_x - v_0; \quad v'_y = v_y; \quad v'_z = v_z$$

Similarly differentiating both side w.r.t t we get $\frac{dv'_x}{dt} = \frac{dv_x}{dt}, \frac{dv'_y}{dt} = \frac{dv_y}{dt}, \frac{dv'_z}{dt} = \frac{dv_z}{dt}$

Then transformation equation for acceleration becomes

$$a'_x = a_x; \quad a'_y = a_y; \quad a'_z = a_z;$$

The transformation rule for partial derivatives operators for Galilean transformation.

[Note that: $If\ f = f(x', t');\ x' = x'(x, t);\ t' = t'(x, t)$ then according to chain rule for partial derivatives $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x}$ & $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial t}$]

As $\frac{\partial x'}{\partial x} = 1;\ \frac{\partial t'}{\partial x} = 0;\ \frac{\partial x'}{\partial t} = -v_0;\ \frac{\partial t'}{\partial t} = 1\ for$ Galilean Transformation. Then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}$ or, $\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$

$$\& \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x'}(-v_0) + \frac{\partial f}{\partial t'} \text{ or, } \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v_0\frac{\partial}{\partial x'} \Longrightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \vec{v}_0.\vec{\nabla}$$

Then $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}$ & $\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \left(\frac{\partial}{\partial t'} - v_0\frac{\partial}{\partial x'}\right)\left(\frac{\partial f}{\partial x'}(-v_0) + \frac{\partial f}{\partial t'}\right) = -v_0\frac{\partial^2 f}{\partial t'\partial x'} + \frac{\partial^2 f}{\partial t'^2} + v_0^2\frac{\partial^2 f}{\partial x'^2} - v_0\frac{\partial^2 f}{\partial t'\partial x'} = \frac{\partial^2 f}{\partial t'^2} + v_0^2\frac{\partial^2 f}{\partial x'^2} - 2v_0\frac{\partial^2 f}{\partial t'\partial x'}$

Hence $\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}$ & $\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t'^2} + v_0^2\frac{\partial^2}{\partial x'^2} - 2v_0\frac{\partial^2}{\partial t'\partial x'}$

Show that 'LENGTH' is invariant under Galilean transformation.

In the **S frame** the distance between two points (x_1, y_1, z_1) & (x_2, y_2, z_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

In the **S' frame** this distance transformed to $\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2} = \sqrt{(x_2 - vt - x_1 + vt)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ = the distance between two points in **S frame**

[Note that Hence the 'AREA' and 'VOLUME' is Galilean invariant]

Show that **'FORCE' is Galilean invariant or, Newton's law** is Galilean invariant

For Galilean transformation $x' = x - v_0 t;\quad y' = y;\quad z' = z;\quad t' = t$ we have $a'_x = a_x;\quad a'_y = a_y;\ a'_z = a_z;$

Hence $\vec{a'} = \vec{a}$ that means acceleration is Galilean invariant

We have $\vec{F} = m\vec{a}$ then $\vec{F'} = m'\vec{a'} = m\vec{a} = \vec{F}$ (proved) [as m is Galilean invariant.]

Show that **CHARGE DENSITY** is Galilean invariant.

Charge density $\rho = \frac{q}{V}$ as q is invariant according to charge conservation theorem and volume is Galilean invariant then charge density Galilean invariant.

Show that **CURRENT DENSITY is not** Galilean invariant.

As current density $\vec{J} = \rho\vec{v}$ as ρ is Galilean invariant but $\vec{v}$ is not as $v'_x = v_x - v_0; v'_y = v_y;\ v'_z = v_z$ i.e, $\vec{v}' = \vec{v} - \vec{v}_0$ then $\vec{J}' = \rho(\vec{v} - \vec{v}_0) = \vec{J} - \rho\vec{v}_0$

Check the invariance of **ELECTRIC FIELD STRENGTH** and **MAGNETIC INDUCTION VECTOR** from the pre-relativistic requirement of invariance of Lorentz force.

We have $\vec{F} = q\{\vec{E} + (\vec{v} \times \vec{B})\}$ then $\vec{F'} = q\{\vec{E'} + (\vec{v'} \times \vec{B'})\} = q\left\{\vec{E'} + \left((\vec{v} - \vec{v}_0) \times \vec{B'}\right)\right\}$

Hence $\vec{E'} + \left((\vec{v} - \vec{v}_0) \times \vec{B'}\right) = \vec{E} + (\vec{v} \times \vec{B})$ or, $\vec{E'} + \vec{v} \times \left(\vec{B'} - \vec{B}\right) = \vec{E} + \vec{v}_0 \times \vec{B'}$

Then the only possible solution of equ^n (i) not involving velocities in specific reference frame, and without restriction on the choice $\vec{v'}$ is $\vec{E'} = \vec{E} + \vec{v}_0 \times \vec{B'}$ & $\vec{B'} = \vec{B}$

Show that MAGNETIC GAUSS'S LAW is Galilean invariant

Here we have $\vec{\nabla'}.\overrightarrow{B'} = \vec{\nabla}.\vec{B} = 0$ hence solenoidal character of $\vec{B}$ is invariant under Galilean transformation.

Show that FARADAY'S LAW is Galilean invariant

$$\vec{\nabla'} \times \overrightarrow{E'} + \frac{\partial \overrightarrow{B'}}{\partial t'} = \vec{\nabla} \times \left(\vec{E} + \vec{v}_0 \times \vec{B}\right) + \frac{\partial \vec{B}}{\partial t} + \left(\vec{v}_0.\vec{\nabla}\right)\vec{B} = \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{v}_0 \times \vec{B}\right) + \left(\vec{v}_0.\vec{\nabla}\right)\vec{B}$$

As $\vec{v}_0$ is a constant vector then $\vec{\nabla} \times \left(\vec{v}_0 \times \vec{B}\right) = \vec{v}_0\left(\vec{\nabla}.\vec{B}\right) - \vec{B}\left(\vec{\nabla}.\vec{v}_0\right) + \left(\vec{B}.\vec{\nabla}\right)\vec{v}_0 - \left(\vec{v}_0.\vec{\nabla}\right)\vec{B} = \mathbf{0} - \mathbf{0} + \mathbf{0} - \left(\vec{v}_0.\vec{\nabla}\right)\vec{B} = -\left(\vec{v}_0.\vec{\nabla}\right)\vec{B}$

$$\vec{\nabla'} \times \overrightarrow{E'} + \frac{\partial \overrightarrow{B'}}{\partial t'} = \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \mathbf{0} \text{ (proved)}$$

Show that GAUSS'S LAW IN ELECTROSTATIC is not Galilean invariant

$$\vec{\nabla'}.\overrightarrow{E'} - \frac{\rho'}{\epsilon_0} = \vec{\nabla}.\left(\vec{E} + \vec{v}_0 \times \overrightarrow{B'}\right) - \frac{\rho}{\epsilon_0} = \vec{\nabla}.\vec{E} - \frac{\rho}{\epsilon_0} + \vec{\nabla}.\left(\vec{v}_0 \times \overrightarrow{B'}\right) = \vec{\nabla}.\left(\vec{v}_0 \times \overrightarrow{B'}\right) \neq \vec{\nabla}.\vec{E} - \frac{\rho}{\epsilon_0} \text{ (PROVED)}$$

Show that AMPERE'S LAW is not Galilean invariant

$$\vec{\nabla'} \times \vec{B'} - \frac{1}{c^2}\frac{\partial \vec{E'}}{\partial t'} - \mu_0\vec{J'} = \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\left(\frac{\partial\left(\vec{E} + \vec{v}_0 \times \vec{B'}\right)}{\partial t} + \left(\vec{v}_0.\vec{\nabla}\right)\left(\vec{E} + \vec{v}_0 \times \vec{B'}\right)\right) - \mu_0\left(\vec{J} - \rho\vec{v}_0\right)$$

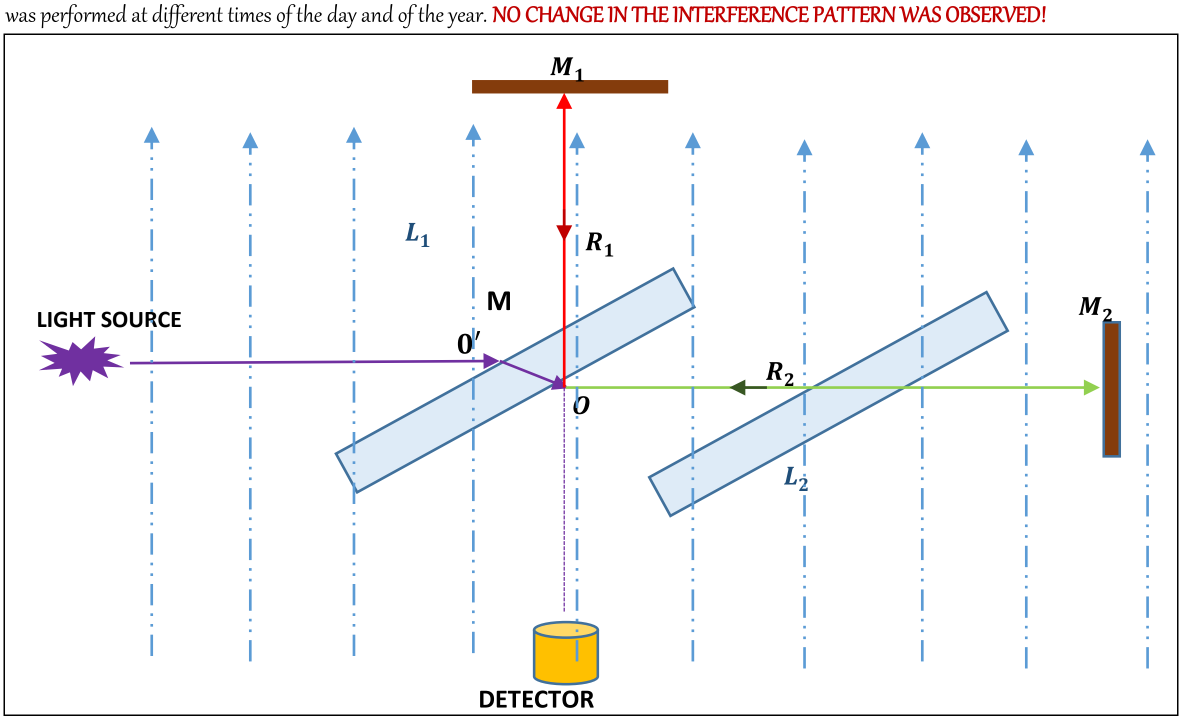
$$= \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \mu_0\vec{J} - \frac{1}{c^2}\vec{v}_0 \times \left[\frac{\partial}{\partial t} + \left(\vec{v}_0.\vec{\nabla}\right)\right]\vec{B} - \frac{1}{c^2}\left(\vec{v}_0.\vec{\nabla}\right)\vec{E} + \mu_0\rho\vec{v}_0 \neq \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \mu_0\vec{J} \text{ (PROVED)}$$

Show that THE CHARGE CONTINUITY EQUATION is invariant under Galilean transformation

$$\vec{\nabla'}.\vec{J'} + \frac{\partial \rho'}{\partial t'} = \vec{\nabla}.\vec{J} - \vec{\nabla}.\rho\vec{v}_0 + \left(\frac{\partial}{\partial t} - \vec{v}_0.\vec{\nabla}\right)\rho = \vec{\nabla}.\vec{J} - \frac{\partial \rho}{\partial t} \text{ (proved)}$$

THE MICHELSON-MORLEY EXPERIMENT

Note that - Sound waves need a medium though which to travel. In 1864 James Clerk Maxwell showed that light is an electromagnetic wave. Therefore it was assumed that there is an ether which propagates light waves. This ether was assumed to be everywhere and unaffected by matter. This ether could be used to determine an absolute reference frame (with the help of observing how light propagates through the ether). Note. The Michelson-Morley experiment (circa 1885) was performed to detect the Earth's motion through the ether as follows: The viewer will see the two beams of light which have travelled along 1 different arms display some interference pattern. If the system is rotated, then the influence of the "ether wind" should change the time the beams of light take to travel along the arms and therefore should change the interference pattern. The experiment was performed at different times of the day and of the year. NO CHANGE IN THE INTERFERENCE PATTERN WAS OBSERVED!



The beam of light from laboratory source S is split by the partially silvered mirror M into two coherent beam $R_1(reflected\ through\ M)$& $R_2(transmitted\ through\ M)$. R_1 reflected back to M by M_1 mirror and R_2 reflected back to M by M_2 mirror. Then this two beam interfere at O. The interference pattern is constructive or destructive depending on the phase (path) difference between the two light. The half silvered mirror is inclined at $45°$ to the beam direction. M_1 & $M_2$are very nearly at right angle. We shall observe a fringe system in the telescope.

<u>Let compute the phase difference between two light beams R_1 & R_2</u>

This phase difference arise due to (i) Different path length (L_1 & L_2) travelled.

(ii) Different speed of travel due to **existence of either wind** (let it's velocity w.r.t laboratory is $\boldsymbol{v}$)

The time taken for beam R_1 from M to $M_1$and back to M is $t_1 = \frac{L_1}{c-v} + \frac{L_1}{c+v} = L_1\left(\frac{2c}{c^2-v^2}\right) = \frac{2L_1}{c}\left(\frac{1}{1-{v^2}/{c^2}}\right)$

The time taken for beam R_2 from M to $M_2$and back to M is $t_2 = \frac{2\left[{L_2}^2+\left(\frac{vt_2}{2}\right)^2\right]^{\frac{1}{2}}}{c}$ then $t_2 = \frac{2L_2}{c}\left(\frac{1}{\sqrt{1-{v^2}/{c^2}}}\right)$

Note that the calculation t_2 is made in either frame and t_1 in laboratory frame. In classical mechanics time is absolute.

The difference in transit time is $\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{L_2}{\sqrt{1-{v^2}/{c^2}}} - \frac{L_1}{1-{v^2}/{c^2}}\right)$

Now suppose the instrument is rotated by an angle $90°$ then $\Delta t' = t'_2 - t'_1 = \frac{2}{c}\left(\frac{L_2}{1-{v^2}/{c^2}} - \frac{L_1}{\sqrt{1-{v^2}/{c^2}}}\right)$

Hence the rotation changes the time difference by $\Delta t'$-$\Delta t = \frac{2}{c}\left(\frac{L_2+L_1}{1-{v^2}/{c^2}} - \frac{L_1+L_2}{\sqrt{1-{v^2}/{c^2}}}\right)$

$$= \frac{2(L_1+L_2)}{c}\left[1 + {v^2}/{c^2} - 1 - \frac{1}{2}{v^2}/{c^2} \dots\dots\right] \approx \frac{(L_1+L_2)}{c}\frac{v^2}{c^2}$$

Therefore rotation should cause a shift in fringe pattern.

Let ΔN represent the number of fringes moving past the crosshairs as the pattern shift. Then $\Delta N = \frac{\Delta t' - \Delta t}{T} \approx \frac{(L_1+L_2)}{cT}\frac{v^2}{c^2} = \frac{(L_1+L_2)}{\lambda}\frac{v^2}{c^2}$

As earth rotating with a speed of 30km/s then if Michelson Morley take $L_1 = L_2 = 22m$ $\lambda = 5.5\times10^{-7}m$ $\frac{v}{c} = 10^{-4}$

Then $\Delta N = 0.4$ or shift of fourth tenths a fringe.

Experimental results: there was no fringe shift at all.

(a) **The principle of equivalence of physical law**: All phenomena of physics appear the same in all uniformly moving frame. This implies that laws of physics are independent of motion (with uniform velocity) of a particular space-time co-ordinate system.

(b) **Principle of constancy of velocity of light**: Speed of light is same in all direction and independent of relative-uniform motion of the observer, source of light and the medium.

LORENTZ TRANSFORMATION:

For this type of transformation the assumptions are **(i) It must be linear. (ii) It should preserved the speed of light. (iii) It should approach the Galilean transformation in the limit of low speeds compared to speed of light**.

We consider two uniform moving system origin of those coincide at $t = 0$. Further at $t = 0$, let the source of light, fixed at the origin of unprimed system, emit a pulse of light.

Thus an observer in an unprimed system will see a spreading spherical wave front propagating with speed c the equation of which can be written as $x^2 + y^2 + z^2 = c^2t^2$………………………..(i)

An observer in a primed system will see a spreading spherical wave front propagating with speed c (as speed of light is invariant) and the equation of which can be written as $x'^2 + y'^2 + z'^2 = c^2t'^2$………………………..(ii)

Note that the need of transformation of time scale in going from one system to another due to the fact that no arrangement of space co-ordinate alone can give wave pulse that are simultaneously concentric in both co-ordinate.

Inspecting of equation (i) & (ii) we reveals that the desired transformation must be such that

$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$ ……………..(iii) or, $\sum_{i=1}^{3} x_i^{\,2} - c^2t^2 = \sum_{i=1}^{3} {x'_i}^2 - c^2t'^2$

If we put $x_4 = ict$ then we have, $\sum_{\mu=1}^{4} x_\mu^{\,2} = \sum_{\mu=1}^{4} {x'_\mu}^2$ **……………………..(iv)**

Basing on analogy of spatial rotation of the co-ordinate axes in 3D orthogonal space, we can say that equation (iv) represent orthogonal transformation of a vector in 4D space.(**we see later that this space is known as Minkowski space or world space.**)

Hence Lorentz transformation are merely the orthogonal transformation in 4D space. Then Lorentz transformation can be represented as $x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu}x_\nu$ **{where $a_{\mu\nu}$ is the linear transformation matrix}**

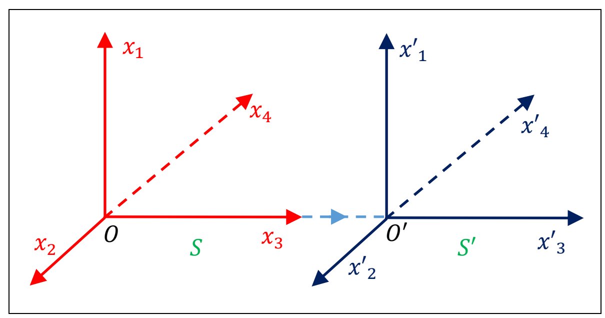
Note that here we consider only pure Lorentz transformation which does not involve any spatial rotation, where General Lorentz transformation is product of space rotation and pure Lorentz transformation.

As $x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu}x_\nu$ then $x'_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$

$x'_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4$

$x'_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4$

$x'_4 = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4$



Note that x_4 is imaginary then a_{14}, a_{24}, a_{34} must be imaginary as x'_1, x'_2, x'_3 are real. Similarly a_{41}, a_{42}, a_{43} will be imaginary and a_{44} real.

Let us consider the primed system is moving with a velocity v along X_3 axis then $x'_1 = x_1, x'_2 = x_2$ then

$x'_1 = x_1$

$x'_2 = x_2$

$x'_3 = a_{33}x_3 + a_{34}x_4$

$x'_4 = a_{43}x_3 + a_{44}x_4$

Then transformation matrix is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}$

We have the orthogonality condition as ,

$$a_{33}^{\,2} + a_{34}^{\,2} = 1, \qquad a_{43}^{\,2} + a_{44}^{\,2} = 1,$$

$$a_{33}a_{43} + a_{34}a_{44} = 0$$

The four unknown elements can be uniquely determined when fourth relation b/w them is provided.

The origin of primed co-ordinate system is moving uniformly along X_3 axis. For unprimed system then $x_3 = vt = v\frac{x_4}{ic} = -i\beta x_4$ [**Where $\beta = \frac{v}{c}$**] but for primed system $x'_3 = a_{33}x_3 + a_{34}x_4 = 0$ then $a_{33}(-i\beta x_4) + a_{34}x_4 = 0$ or, $a_{34} = i\beta a_{33}$

Then $a_{33} = \frac{1}{\sqrt{1-\beta^2}}$, $a_{34} = \frac{i\beta}{\sqrt{1-\beta^2}}$ then $\frac{1}{\sqrt{1-\beta^2}} a_{43} + \frac{i\beta}{\sqrt{1-\beta^2}} a_{44} = 0$ or, $a_{43} = -i\beta a_{44}$ then $a_{44} = \frac{1}{\sqrt{1-\beta^2}}$, $a_{43} = \frac{-i\beta}{\sqrt{1-\beta^2}}$

then $x'_1 = x_1$; $x'_2 = x_2$; $x'_3 = \frac{1}{\sqrt{1-\beta^2}}x_3 + \frac{i\beta}{\sqrt{1-\beta^2}}x_4$ & $x'_4 = \frac{-i\beta}{\sqrt{1-\beta^2}}x_3 + \frac{1}{\sqrt{1-\beta^2}}x_4$

putting $x_1 = x, x_2 = y, x_3 = z, x_4 = ict$ & $x'_1 = x', x'_2 = y', x'_3 = z', x'_4 = ict'$ **we get**

$$\left.\begin{aligned} x' &= x \\ y' &= y \\ z' &= \frac{z - vt}{\sqrt{1-\beta^2}} \\ t' &= \frac{t - \frac{v}{c^2}z}{\sqrt{1-\beta^2}} \end{aligned}\right\} \text{ LORENTZ TRANSFORMATION}$$

Show that Lorentz transformation may be regarded as a rotation of axes through an imaginary angle.

Let us consider a four dimensional a four dimensional space with dimensions X_1, X_2, X_3 & X_4. When $X_4 = ict$. Suppose X_1 & X_4 axes are rotated through an angle θ. So new axes are X'_1 & X'_4, X_2 & X_3 axes are remain unchanged.

Then obviously, $X'_1 = X_1 \cos\theta + X_4 \sin\theta,\ X'_2 = X_2,\ X'_3 = X_3\ \&\ X_4 = -X_1 \sin\theta + X_4 \cos\theta$

Let us take angle θ be imaginary such that $\tan\theta = i\beta$. Then $\sin\theta = \frac{i\beta}{\sqrt{1-\beta^2}}$ and $\cos\theta = \frac{1}{\sqrt{1-\beta^2}}$

Then $X'_1 = X_1 \frac{1}{\sqrt{1-\beta^2}} + X_4 \frac{i\beta}{\sqrt{1-\beta^2}} = \frac{X_1 + ict\cdot i\beta}{\sqrt{1-\beta^2}} = \frac{X_1 - vt}{\sqrt{1-\beta^2}}$ and $X_4 = -X_1 \frac{i\beta}{\sqrt{1-\beta^2}} + X_4 \frac{1}{\sqrt{1-\beta^2}}$ $\therefore ict' = -\frac{i\beta X_1}{\sqrt{1-\beta^2}} + \frac{ict}{\sqrt{1-\beta^2}} \Rightarrow t' = \frac{t - \frac{v}{c^2}X_1}{\sqrt{1-\beta^2}}$

Then we have, $\boldsymbol{X'_1 = \frac{X_1 - vt}{\sqrt{1-\beta^2}},\ X'_2 = X_2,\ X'_3 = X_3\ \&\ t' = \frac{t - \frac{v}{c^2}X_1}{\sqrt{1-\beta^2}}}$ (Proved)

Prove that four dimensional volume element "dx, dy, dz, dt" is invariant under Lorentz transformation.

We have Lorentz transformation when primed system initially coincide with unprimed system moving with a velocity(v) along x-axis. $X' = \frac{X - vt}{\sqrt{1-\beta^2}}, Y' = Y, Z' = Z\ \ \&\ t' = \frac{t - \frac{v}{c^2}X}{\sqrt{1-\beta^2}}$.

Then for any two points the inverse Lorentz transformation $X_1 = \frac{X'_1 + vt'}{\sqrt{1-\beta^2}}$ and $X_2 = \frac{X'_2 + vt'}{\sqrt{1-\beta^2}}$

[Here we consider simultaneous measurement of X'_1 & X'_2 i.e. $t'_1 = t'_2 = t'$ or, X_1 & X_2 i.e. $t_1 = t_2 = t$]

Then we have $X_2 - X_1 = \frac{X'_2 - X'_1}{\sqrt{1-\beta^2}}$ $or, dx = \frac{dx'}{\sqrt{1-\beta^2}} \Rightarrow dx' = dx\sqrt{1-\beta^2}$ [Lorentz-Fitzgerald construction]

Now the time interval between two event happening in the same space point.

$t'_2 - t'_1 = \frac{t_2 - \frac{v}{c^2}X}{\sqrt{1-\beta^2}} - \frac{t_1 - \frac{v}{c^2}X}{\sqrt{1-\beta^2}}$ i.e. $dt' = \frac{dt}{\sqrt{1-\beta^2}}$ [Time dilation]

As the motion of primed system is along X-axis then $dy' = dy$ and $dz' = dz$.

Then $dx'dy'dz'dt' = dx\sqrt{1-\beta^2}dydz\frac{dt}{\sqrt{1-\beta^2}} = dxdydzdt$.

- **For moving observer, length of any rod appears to be contracted this is known as Lorentz-Fitzgerald construction.**
- **It appears to the moving observer that the stationary clock is moving at slow rate. The effect is called time dilation.**

In the above discussion, note that $dx' < dx$ but $dt' > dt$.

Proper length: The proper length of an object is defined as the length measured by a scale at rest with respect to object. Note that measurement of two end points must be simultaneous.

Proper time: The time interval between two events which occur at the same position measured by a clock in the inertial frame in which the event occur is called proper time.

Explain the 'simultaneity' of two events in relativity. Discuss 'order of event'.

Consider two events characterized by the space and time coordinates (x_1, y_1, z_1, t_1) & (x_2, y_2, z_2, t_2) in S-frame and (x'_1, y'_1, z'_1, t'_1) & (x'_2, y'_2, z'_2, t'_2) in S'-frame. Let at t=0 the two frames coincide and S'-frame is moving with a velocity $v\hat{\imath}$ relative to S-frame i.e. along x-axis.

The two events are simultaneous in S-frame if $t_1 = t_2$ and two events are simultaneous in S'-frame if $t'_2 = t'_1$.

Order of event means the sequence in which the events occur in any frame.

Note that, (a) Two events which are simultaneous in S'-frame.

As $t'_2 = \frac{t_2 - \frac{v}{c^2}x_2}{\sqrt{1-\beta^2}}$ and $t'_1 = \frac{t_1 - \frac{v}{c^2}x_1}{\sqrt{1-\beta^2}}$ Then, $t'_2 - t'_1 = \frac{1}{\sqrt{1-\beta^2}}\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$…………(i)

If two events are simultaneous in S-frame then $t_2 = t_1$ at different position $(x_2 \neq x_1)$ then$(t'_2 - t'_1) \neq 0$, i.e. they are not simultaneous in S'-frame.

(b) Two simultaneous event at different position differ by (Δx) in S-frame, the time interval between the two events in S'-frame is then from eq(i), $\Delta t' = -\frac{\frac{v}{c^2}\Delta x}{\sqrt{1-\beta^2}}$.

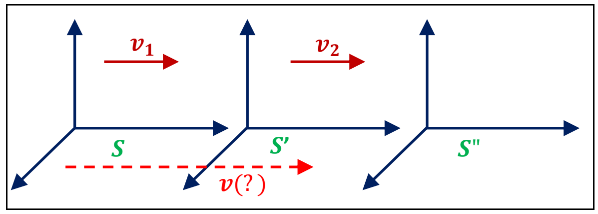
(c) The order of events shall remain in the same in both the inertial frame.

We have from eq(i) when $(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1) = 0$ then events are simultaneous in S'-frame.

If $(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1) > 0$ then order of event is same in both frame. When $\frac{v}{c^2}(x_2 - x_1) > (t_2 - t_1)$ then S'-frame will observe the reverse order. But for that as v is always less than c then $\frac{1}{c}(x_2 - x_1) > (t_2 - t_1) \Rightarrow (x_2 - x_1) > c(t_2 - t_1)$ is possible when signal can travel with a velocity greater than c. Hence reverse order of event is impossible.

Show that Lorentz transformation carried out in succession are equivalent to one Lorentz transformation.

Let us consider three frame of reference S, S' & S''. S' is moving w.r.t S with a velocity v_1 along Z-axis and S'' is moving w.r.t S' with a velocity v_2 along Z-axis. We want to find out velocity of S'' w.r.t S.



Then $x' = x\,, y' = y, z' = \dfrac{z - v_1 t}{\sqrt{1-\frac{{v_1}^2}{c^2}}}, t' = \dfrac{t - \frac{v_1}{c^2}z}{\sqrt{1-\frac{{v_1}^2}{c^2}}}$....................(i)

$x'' = x'\,, y'' = y', z'' = \dfrac{z' - v_2 t'}{\sqrt{1-\frac{{v_2}^2}{c^2}}}, t'' = \dfrac{t' - \frac{v_2}{c^2}z'}{\sqrt{1-\frac{{v_2}^2}{c^2}}}$.................(ii)

From equation (i)&(ii) we have,

$$x'' = x\,, y'' = y, z'' = \frac{\frac{z-v_1 t}{\sqrt{1-\frac{{v_1}^2}{c^2}}} - v_2\left(\frac{t-\frac{v_1}{c^2}z}{\sqrt{1-\frac{{v_1}^2}{c^2}}}\right)}{\sqrt{1-\frac{{v_2}^2}{c^2}}} = \frac{z\left(\frac{1+\frac{v_1 v_2}{c^2}}{\sqrt{1-\frac{{v_1}^2}{c^2}}}\right) + t\left(\frac{(v_1+v_2)}{\sqrt{1-\frac{{v_1}^2}{c^2}}}\right)}{\sqrt{1-\frac{{v_2}^2}{c^2}}} = \frac{\left\{z - t\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right)\right\}\left(1+\frac{v_1 v_2}{c^2}\right)}{\sqrt{\left(1-\frac{{v_1}^2}{c^2}\right)\left(1-\frac{{v_2}^2}{c^2}\right)}} = \frac{\left[z - t\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right)\right]}{\frac{1}{\left(1+\frac{v_1 v_2}{c}\right)}\left[\sqrt{\left\{\left(1+\frac{v_1 v_2}{c^2}\right)^2 - \frac{(v_1+v_2)^2}{c^2}\right\}}\right]} = \frac{z - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

[Where $v = \left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right)$] $\Rightarrow$ **This is relativistic law of addition of velocity.** Similar $t'' = \dfrac{t - \frac{v}{c^2}z}{\sqrt{1-\frac{v^2}{c^2}}}$.

Note that the relativistic law of addition of velocity assures that it is impossible to combine several Lorentz transformation in one involving a relative velocity greater than c.

[As if suppose $v_1 = c$ and $v_2 = c$ then $v = \frac{2c}{1+\frac{c^2}{c^2}} = c$]

Show that under the Lorentz transformation the wave equation for the propagation of electromagnetic potential $\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{dt}\phi = 0$ remains invariant form.

$x' = x\,,\quad y' = y,\quad z' = \frac{z - vt}{\sqrt{1-\beta^2}} = \nu(z - vt),\quad t' = \nu\left(t - \frac{v}{c^2}z\right)$

or, $\quad x = x',\quad y = y',\quad z = \nu(z' + vt'),\quad t = \nu\left(t' + \frac{v}{c^2}z'\right)\quad$ [Where $\nu = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$]

Which gives, $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'},\ \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}\frac{\partial y}{\partial y'}\ ,\ \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}\frac{\partial z}{\partial z'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial z'}\ \ and\ \ \frac{\partial}{\partial t'} = \frac{\partial}{\partial z}\frac{\partial z}{\partial t'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial t'}$

Now, $\frac{\partial x}{\partial x'} = 1, \frac{\partial y}{\partial y'} = 1, \frac{\partial z}{\partial z'} = \nu \Rightarrow \frac{\partial z}{\partial t'} = \nu v, \frac{\partial t}{\partial t'} = \nu \Rightarrow \frac{\partial t}{\partial t'} = \nu\frac{v}{c^2}$

$\therefore \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \frac{\partial}{\partial z'} = \nu\left(\frac{\partial}{\partial z} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$ & $\frac{\partial}{\partial t'} = \nu\left(v\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)$

Then, $\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu^2\left(\frac{\partial}{\partial z} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)^2 - \frac{1}{c^2}\nu^2\left(v\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)^2$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu^2\left[\left(\frac{\partial^2}{\partial z^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial z\partial t} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2}\right) - \frac{1}{c^2}\left(v^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial z\partial t}\right)\right]$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu^2\left[\frac{\partial^2}{\partial z^2}\left(1 - \frac{v^2}{c^2}\right) + 0 - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu^2\left(1 - \frac{v^2}{c^2}\right)\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\right)\right]$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\right) = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box^2 \ \Rightarrow\ \textbf{D'Alemberts Operator}$$

Then $\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{dt}\phi = 0$ remains inverted as ϕ is scalar.

(A) World velocity or Four velocity: World velocity is defined as the rate of change of the position vector of a particle w.r.t its proper time i.e. $u_\mu = \frac{dx_\mu}{d\tau}$. With respect to space coordinate $u_i = \frac{dx_i}{dt\sqrt{1-\beta^2}} = \frac{x_i}{\sqrt{1-\beta^2}}$ and w.r.t time component $u_\phi = \frac{icdt}{d\tau} = \frac{ic}{\sqrt{1-\beta^2}}$.

The magnitude of world velocity is, $u_\mu u_\mu = \left[\left(\frac{v_i}{\sqrt{1-\beta^2}}\right)^2 + \left(\frac{ic}{\sqrt{1-\beta^2}}\right)^2\right] = \frac{v^2}{1-\beta^2} - \frac{c^2}{1-\beta^2} = -\frac{c^2\left(1-\frac{v^2}{c^2}\right)}{1-\beta^2}$.

Since square of world velocity is less than zero, it is time like and has constant magnitude. Thus ${u_M}^2$ is invariant under Lorentz transformation.

(B) Four force and Four momentum: Consider Newton's equation $F_i = \frac{d}{dt}(mv_i)$ which is not invariant under Lorentz transformation. (i) Since t is not Lorentz invariant, it should be replaced by proper time τ. (ii) m can be taken as invariant property of the particle. (iii) In place of v_i, world velocity u_μ should be substituted. (iv) The left hand side F_i should be replaced some four vector K_μ, may be called Minkowski force.

Then Relativistic generalization of Newton's Law becomes, $\frac{d}{d\tau}(mu_\mu) = K_\mu$

For spatial part, $\frac{d}{dt}\frac{1}{\sqrt{1-\beta^2}}\left(\frac{mv_i}{\sqrt{1-\beta^2}}\right) = K_i \;\Rightarrow\; \frac{d}{d\tau}\left(\frac{mv_i}{\sqrt{1-\beta^2}}\right) = K_i\sqrt{1-\beta^2}$

If we use classical definition of force and define force as time rate of change of momentum in all Lorentz system, then classical force should be defined as, $F_i = \frac{d}{d\tau}\left(\frac{mv_i}{\sqrt{1-\beta^2}}\right)$. Then space component of K_μ is $K_i = \frac{F_i}{\sqrt{1-\beta^2}}$ and relativistic definition of classical linear momentum of the particle $P_i = \frac{mv_i}{\sqrt{1-\beta^2}}$.

Now the dot product of four vector K_μ and world velocity u_μ is $K_\mu u_\mu = K_i\frac{v_i}{\sqrt{1-\beta^2}} + K_4\frac{ic}{\sqrt{1-\beta^2}} = 0$

$$\therefore \frac{F_i v_i}{\sqrt{1-\beta^2}} + K_4\frac{ic}{\sqrt{1-\beta^2}} = 0 \Rightarrow \frac{\vec{F}\cdot\vec{v}}{\sqrt{1-\beta^2}} = -icK_4 \qquad \therefore \boldsymbol{K_4 = \frac{i}{c}\frac{\vec{F}\cdot\vec{v}}{\sqrt{1-\beta^2}}}.$$

(C) <u>Relativistic Kinetic Energy:</u> When T=K.E then, $K_4 = \frac{i}{c}\frac{dT}{dt}\frac{1}{\sqrt{1-\beta^2}}$ …………(i)

$$\text{Again, } K_4 = \frac{d}{d\tau}(mu_4) = -\frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left(\frac{mic}{\sqrt{1-\beta^2}}\right)\text{……………………(ii)}$$

From eq(i) & (ii) we have, $\frac{i}{c}\frac{1}{\sqrt{1-\beta^2}}\frac{dT}{dt} = -\frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left(\frac{mic}{\sqrt{1-\beta^2}}\right) \Rightarrow \frac{dT}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\beta^2}}\right)$

Then $T = \frac{mc^2}{\sqrt{1-\beta^2}} + T_0$. Where $T_0 = Constant\ of\ integration.$

For small velocities $(1-\beta^2)^{-\frac{1}{2}}$ can be written as $\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$.

Hence$T = mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) + T_0$. Then for classical K.E $T_0 = -mc^2$. We prefer the choice $T_0 = 0$, then $K_4 = \frac{d}{d\tau}\left(\frac{iT}{c}\right)$.

Then fourth component of momentum is $P_4 = \frac{iT}{c}$.

Now, $P_\mu P_\mu = \left(\frac{mu_i}{\sqrt{1-\beta^2}}\right)^2 + \left(\frac{iT}{c}\right)^2 = \frac{m{v_i}^2}{(1-\beta^2)} - \left(\frac{T}{c}\right)^2 = P^2 - \frac{T^2}{c^2}$. Again $P_\mu P_\mu = m^2{u_\mu}^2 = -m^2c^2$

Then $-m^2c^2 = P^2 - \frac{T^2}{c^2} \qquad \therefore T^2 = P^2c^2 + m^2c^4$

(D) <u>Relativistic Mass:</u> We can preserve the classical definition of linear momentum as the product of mass time the velocity of the particle by defining mass m_r of a particle to be observed to be moving with speed v, by $m_r = \frac{m_0}{\sqrt{1-\beta^2}}$ so that the four momentum $\vec{P} = m_r\vec{v}$. Where m_r is called relativistic mass of the particle where m_0 is called the rest mass of the particle and is scalar invariant. Then Newton's 2nd law becomes, $\vec{F} = \frac{d}{dt}(\vec{P}) = \frac{d}{dt}(m_r\vec{v}) = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0\frac{d}{dt}\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$

If the velocity varies with time then, $\boldsymbol{F} = m_0\left[\frac{du}{dt}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + v\left\{-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v}{c^2}\right)\right\}\frac{dv}{dt}\right] = m_0\frac{dv}{dt}\left[\frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\right] = \boldsymbol{m_0\frac{dv}{dt}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}}$

(E) <u>Mass-Energy Relation:</u> Taking account variation of mass with velocity we write,

$$T = \int_0^r \vec{F}\cdot d\vec{r} = \int_0^r \dot{\vec{P}}\cdot\frac{d\vec{r}}{dt}dt = \int_0^t \dot{\vec{P}}\cdot\vec{v}dt = \int_0^P vdp = \int_0^v vd\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

Integrating by parts we get, $T = \frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\int_0^v\frac{vdv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0c^2\left[\sqrt{1-\frac{v^2}{c^2}}\right]_0^v \Rightarrow T = \frac{m_0v^2+m_0c^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$

$\therefore T = m_rc^2 - m_0c^2 = (m_r - m_0)c^2 = \Delta mc^2$

Thus the K.E of the particle is equal to the product of the increase in mass, Δm and square of the speed of light. Note that here $T_0 = -m_0c^2$. Then $T = \frac{m_0c^2}{\sqrt{1-\beta^2}} + T_0$. Which we have already obtained.