Semester I
DS-1
Mathematical Methods-I (Theory)
PHSDSC101T
(Calculus)
Tutorial lec-13-18

DEPARTMENT OF PHYSICS, P.R.T.G.C

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Contents

1 Questions (Each approximately 2 marks)

Q1: Definition of Limit:
Question: Define the limit of a function $f(x)$ at a point a.
Answer: The limit L of $f(x)$ as $x \to a$ is defined by: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Q2: Continuity vs. Differentiability:
Question: Give an example of a function that is continuous everywhere but not differentiable at some point.
Answer: $f(x) = |x|$ is continuous for all x but not differentiable at $x = 0$.

Q3: Definition of Derivative:
Question: State the definition of the derivative of a function $f(x)$ at $x = a$.
Answer:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Q4: First Principle Differentiation:
Question: Find the derivative of $f(x) = x^2$ using first principles.
Answer:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = 2x.$$

Q5: Taylor Series Statement:
Question: Write the Taylor series expansion for a function $f(x)$ about $x = a$ (state the general form).
Answer:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots .$$

Q6: Binomial Series Statement:
Question: State the binomial series expansion for $(1+x)^n$.
Answer:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots, \quad (|x| < 1 \text{ if } n \notin \mathbb{N}).$$

Q7: Convergence Test:
Question: State one convergence test for an infinite series.
Answer: The Ratio Test: If

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then the series $\sum a_n$ converges absolutely.

Q8: Integrating Factor (ODE):

Question: Define the integrating factor for a first–order linear differential equation.

Answer: For

$$\frac{dy}{dx} + P(x)y = Q(x),$$

an integrating factor is

$$\mu(x) = \exp\left(\int P(x)dx\right)$$

which, when multiplied through, makes the left side an exact derivative.

Q9: Simple First–Order ODE:

Question: Solve $\frac{dy}{dx} + y = e^x$.

Answer: With $\mu(x) = e^x$:

$$\frac{d}{dx}(e^x y) = e^{2x} \quad \Longrightarrow \quad e^x y = \frac{1}{2}e^{2x} + C, \quad y = \frac{1}{2}e^x + C\,e^{-x}.$$

Q10: Existence & Uniqueness Theorem:

Question: State the Existence and Uniqueness Theorem for an initial value problem.

Answer: If $f(x, y)$ and $\partial f/\partial y$ are continuous in a rectangle containing (x_0, y_0), then the IVP

$$y' = f(x, y), \quad y(x_0) = y_0,$$

has a unique solution in some interval around x_0.

Q11: Definition of the Wronskian:

Question: What is the Wronskian of two functions $y_1(x)$ and $y_2(x)$?

Answer:

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

Q12: Exact Differential Equation Criterion:

Question: State the condition for the differential equation $M(x, y)dx + N(x, y)dy = 0$ to be exact.

Answer: The equation is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Q13: Definition of Partial Derivative:

Question: Define the partial derivative of $f(x, y)$ with respect to x.

Answer: It is the derivative of $f(x, y)$ with respect to x while keeping y constant.

Q14: Partial Differentiation Example:

Question: Compute $\frac{\partial}{\partial x}[x^2 y + y^3]$.

Answer:

$$\frac{\partial}{\partial x}(x^2 y) = 2xy, \quad \frac{\partial}{\partial x}(y^3) = 0, \quad \text{so the answer is } 2xy.$$

Q15: Fundamental Trigonometric Limit:

Question: Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

Answer: The limit is 1.

Q16: Taylor Series Convergence Condition:

Question: State the condition for a Taylor series to converge to the function.

Answer: The remainder $R_n(x) \to 0$ as $n \to \infty$ for x in some interval about the expansion point.

Q17: Differentiation of a Product Function:

Question: Differentiate $f(x) = e^x \sin x$.

Answer:

$$f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x).$$

Q18: Slope of a Tangent Line:

Question: Find the slope of the tangent to $y = \ln x$ at $x = 1$.

Answer: $y' = \frac{1}{x}$, so at $x = 1$ the slope is 1.

Q19: Differentiability of $|x|$:

Question: Is $f(x) = |x|$ differentiable at $x = 0$? Explain.

Answer: No; the left–derivative is -1 and the right–derivative is $+1$.

Q20: General Solution of a Homogeneous ODE:

Question: Write the general solution of a homogeneous second–order linear differential equation with constant coefficients.

Answer:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x},$$

where r_1 and r_2 are the roots of the characteristic equation.

Q21: Solve a Simple Homogeneous ODE:

Question: Solve $y'' - 3y' + 2y = 0$.

Answer: The characteristic equation $r^2 - 3r + 2 = 0$ factors as $(r-1)(r-2) = 0$; hence,

$$y = C_1 e^x + C_2 e^{2x}.$$

Q22: Particular Integral Concept:

Question: What is a "particular integral" in solving a nonhomogeneous differential equation?

Answer: It is a specific solution that satisfies the nonhomogeneous equation (without the arbitrary constants).

Q23: Integrating Factor in Exactness:

Question: Explain briefly what an integrating factor does in making a differential form exact.

Answer: It is a function $\mu(x, y)$ that, when multiplied by $Mdx + Ndy$, makes $\mu M dx + \mu N dy$ exact (i.e. its mixed partials become equal).

Q24: Exact Differential Condition (Restated):
Question: What condition must $M(x,y)dx + N(x,y)dy$ satisfy to be exact?
Answer:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Q25: Nonzero Wronskian:
Question: What does a nonzero Wronskian indicate about two functions?
Answer: They are linearly independent.

Q26: Instantaneous Rate of Change:
Question: Define the instantaneous rate of change of a function.
Answer: It is the derivative $f'(x)$, representing the limit of the average rate of change as the interval approaches zero.

Q27: Continuity at a Point:
Question: Define continuity of a function $f(x)$ at $x = a$.
Answer: f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

Q28: Example of Discontinuity:
Question: Provide an example of a function that is discontinuous at a point.
Answer: $f(x) = \frac{1}{x}$ is discontinuous at $x = 0$.

Q29: Differentiability Implies Continuity:
Question: What does differentiability imply about continuity?
Answer: If $f(x)$ is differentiable at a point, then it is continuous there.

Q30: Binomial Theorem (Non–integer Exponent):
Question: State the binomial theorem for a non–integer exponent n.
Answer:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad (|x| < 1).$$

Q31: Derivative of $\sin x$:
Question: Differentiate $f(x) = \sin x$.
Answer: $f'(x) = \cos x$.

Q32: Integrating Factor for Linear ODE:
Question: What is the integrating factor for the linear ODE $\frac{dy}{dx} + P(x)y = Q(x)$?
Answer: $\mu(x) = \exp\left(\int P(x)dx\right)$.

Q33: Separable Equation Example:
Question: Solve $\frac{dy}{dx} = \frac{2x}{y}$ by separation of variables.
Answer: Rearranging, $y\,dy = 2x\,dx$. Integrate: $\frac{1}{2}y^2 = x^2 + C$ or $y^2 = 2x^2 + C'$.

Q34: Inexact Differential:
Question: Define what is meant by an inexact differential.
Answer: A differential form that is not the total differential of any function is called inexact.

Q35: Exactness Criterion (again):
Question: State the criterion for a differential form in two variables to be exact.
Answer: The form $Mdx + Ndy$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Q36: Example of a Differentiable Function:
Question: Give an example of a function that is differentiable everywhere.
Answer: $f(x) = e^x$ is differentiable for all x.

Q37: Instantaneous Velocity:
Question: Explain what is meant by instantaneous velocity.
Answer: It is the derivative of the position function with respect to time.

Q38: Derivative of $\ln x$:
Question: What is the derivative of $\ln x$?
Answer: $\frac{d}{dx} \ln x = \frac{1}{x}$.

Q39: Derivative of $\cos x$:
Question: Differentiate $f(x) = \cos x$.
Answer: $f'(x) = -\sin x$.

Q40: Homogeneous ODE General Solution:
Question: What is the general solution to a homogeneous second–order linear differential equation (with distinct real roots)?
Answer: $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$, where r_1 and r_2 are the roots.

Q41: Particular Integral Concept (Restated):
Question: Explain briefly what is meant by a "particular integral."
Answer: It is a specific solution that satisfies the nonhomogeneous differential equation without the arbitrary constants.

Q42: Meaning of the Existence Theorem:
Question: What does the existence theorem guarantee for an IVP?
Answer: If $f(x, y)$ and $\partial f/\partial y$ are continuous near (x_0, y_0), then a unique solution exists in some interval around x_0.

Q43: Partial Derivative with Respect to y:
Question: Define the partial derivative of $f(x, y)$ with respect to y.
Answer: It is the derivative of $f(x, y)$ with respect to y while treating x as constant.

Q44: Second Partial Derivative Example:
Question: Find the second partial derivative $\frac{\partial^2}{\partial x^2}(x^2 y)$.
Answer: First, $\frac{\partial}{\partial x}(x^2 y) = 2xy$; then, $\frac{\partial^2}{\partial x^2}(x^2 y) = 2y$.

Q45: Separation of Variables (Method Description):
Question: Describe briefly the method of separation of variables for solving differential equations.
Answer: It involves rewriting the ODE so that all terms in x are on one side and those in y on the other, and then integrating both sides.

Q46: Uniform Convergence Test:

Question: State one test for uniform convergence of a series.

Answer: The Weierstrass M–test: if $|f_n(x)| \leq M_n$ for all x and $\sum M_n$ converges, then $\sum f_n(x)$ converges uniformly.

Q47: Implicit Differentiation:

Question: Differentiate implicitly the equation $x^2 + y^2 = 1$ with respect to x.

Answer: Differentiating gives $2x + 2y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{x}{y}$.

Q48: Variation of Parameters (Statement):

Question: State the method of variation of parameters in brief.

Answer: It is a technique to find a particular solution of a nonhomogeneous linear ODE using the known homogeneous solutions by allowing the constants to be functions.

Q49: Singular Point Definition:

Question: Define a singular point in the context of differential equations.

Answer: A point where the coefficient of the highest derivative either vanishes or is not analytic is called a singular point.

Q50: Repeated Roots General Solution:

Question: Write the general solution for a second–order linear differential equation with constant coefficients that has a repeated real root r.

Answer:

$$y(x) = (C_1 + C_2 x)e^{rx}.$$

2 Questions (Each approximately 3–4 marks)

Q1: Limit Evaluation (Factorization):
Question: Evaluate

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

Answer: Factor the numerator:

$$\frac{(x-2)(x+2)}{x-2} = x + 2,$$

so the limit is 4.

Q2: Differentiability Implies Continuity Proof:
Question: Prove that if a function is differentiable at a point, then it is continuous at that point.
Answer: If f is differentiable at a, then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, which implies

$$\lim_{h \to 0} [f(a+h) - f(a)] = 0.$$

Thus, $\lim_{x \to a} f(x) = f(a)$ and f is continuous at a.

Q3: L'Hôpital's Rule Application:
Question: Use L'Hôpital's rule to compute

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$$

Answer: Differentiate numerator and denominator:

$$\lim_{x \to 0} \frac{e^x - 1}{2x}.$$

This is still $\frac{0}{0}$, so differentiate again:

$$\lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.$$

Q4: Quotient Rule Differentiation:
Question: Find the derivative of

$$f(x) = \frac{x^2 + 1}{x - 1}$$

using the quotient rule. **Answer:**

$$f'(x) = \frac{(2x)(x-1) - (x^2+1)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

Q5: Taylor Series for e^x:

Question: Expand e^x in a Taylor series about $x = 0$ up to the x^3 term. **Answer:**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Q6: Taylor Series for $\cos x$:

Question: Derive the Taylor series expansion for $\cos x$ about 0. **Answer:**

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots .$$

Q7: Radius of Convergence for $\ln(1 + x)$:

Question: Determine the radius of convergence for the Taylor series of $\ln(1 + x)$. **Answer:** The series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

converges for $|x| < 1$; hence, the radius is 1.

Q8: First–Order Linear ODE:

Question: Solve the ODE

$$\frac{dy}{dx} + 2y = 4e^{-2x}.$$

Answer: The integrating factor is $\mu(x) = e^{2x}$. Then,

$$\frac{d}{dx}(e^{2x}y) = 4 \quad \Rightarrow \quad e^{2x}y = 4x + C, \quad y = e^{-2x}(4x + C).$$

Q9: Homogeneous Equation via Substitution:

Question: Find the general solution of

$$\frac{dy}{dx} = \frac{y - x}{y + x}.$$

Hint: Use the substitution $v = \frac{y}{x}$. **Answer:** With $y = vx$, we have $y' = v + xv'$. Substituting yields:

$$v + xv' = \frac{v - 1}{v + 1}.$$

Rearrange and separate variables:

$$x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v.$$

After algebra and integration, the implicit solution is:

$$\frac{1}{2} \ln \left|v^2 + 1\right| + \arctan v = -\ln |x| + C,$$

with $v = \frac{y}{x}$.

Q10: Second–Order Homogeneous ODE:

Question: Solve the ODE
$$y'' + 4y' + 4y = 0.$$

Answer: The characteristic equation is
$$r^2 + 4r + 4 = (r + 2)^2 = 0,$$
so
$$y = (C_1 + C_2 x)e^{-2x}.$$

Q11: Wronskian Calculation:

Question: Find the Wronskian of $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$. **Answer:** Compute:
$$y_1' = -2e^{-2x}, \quad y_2' = e^{-2x} - 2xe^{-2x}.$$

Then,
$$W = y_1 y_2' - y_2 y_1' = e^{-2x}(e^{-2x} - 2xe^{-2x}) - xe^{-2x}(-2e^{-2x}) = e^{-4x}.$$

Q12: Method of Undetermined Coefficients:

Question: Solve
$$y'' - y = e^x.$$

Answer: The homogeneous solution is $y_h = C_1 e^x + C_2 e^{-x}$. Since e^x is a solution of the homogeneous part, try $y_p = Ax\,e^x$. Substitution gives $A = \frac{1}{2}$; hence,
$$y = y_h + \frac{1}{2}xe^x.$$

Q13: Exact Equation Verification:

Question: Verify that the differential equation
$$(3x^2 + 2y)\,dx + (2x + 4y)\,dy = 0$$
is exact. **Answer:** Compute:
$$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2.$$

Since they are equal, the equation is exact.

Q14: Exactness with Trigonometric Terms:

Question: For $M(x, y) = 2xy + \cos y$ and $N(x, y) = x^2 - x \sin y$, check if $Mdx + Ndy = 0$ is exact; if not, suggest an integrating factor. **Answer:**
$$M_y = 2x - \sin y, \quad N_x = 2x - \sin y.$$

They are equal, so the equation is already exact.

Q15: Partial Derivatives of a Composite Function:
Question: Compute $\frac{\partial}{\partial x}[x^2y + \sin(xy)]$. **Answer:**

$$\frac{\partial}{\partial x}(x^2y) = 2xy, \quad \frac{\partial}{\partial x}\sin(xy) = y\cos(xy).$$

Thus, the answer is $2xy + y\cos(xy)$.

Q16: Second–Order Partial Derivative:
Question: Compute $\frac{\partial^2}{\partial x^2}e^{xy}$. **Answer:** First,

$$\frac{\partial}{\partial x}e^{xy} = ye^{xy}, \quad \frac{\partial^2}{\partial x^2}e^{xy} = y^2e^{xy}.$$

Q17: Existence and Uniqueness Verification:
Question: Verify the existence and uniqueness theorem for the IVP

$$y' = y + \sin x, \quad y(0) = 1.$$

Answer: Since $f(x, y) = y + \sin x$ and $\frac{\partial f}{\partial y} = 1$ are continuous everywhere, a unique solution exists near $x = 0$.

Q18: IVP for a Second–Order ODE:
Question: Solve the IVP

$$y'' + y = 0, \quad y(0) = 2, \quad y'(0) = 0.$$

Answer: The characteristic equation $r^2 + 1 = 0$ gives $r = \pm i$, so

$$y = C_1\cos x + C_2\sin x.$$

Using $y(0) = 2$ implies $C_1 = 2$ and $y'(0) = C_2 = 0$; hence, $y = 2\cos x$.

Q19: Integrating Factor for a Non–Exact Equation:
Question: Find an integrating factor (if needed) and solve the differential equation

$$(2xy + y^2)dx + (x^2 + 2xy)dy = 0.$$

Answer: Check: $M_y = 2x + 2y$ and $N_x = 2x + 2y$; since they are equal, the equation is exact. One then finds the potential function $\phi(x, y)$ and obtains the solution $\phi(x, y) = C$.

Q20: Homogeneity Check and Solution:
Question: Show that

$$y' = \frac{2x + 3y}{4x - 2y}$$

is homogeneous and solve it. **Answer:** Both numerator and denominator are of degree 1. With $v = y/x$, substitute $y = vx$ and rewrite the ODE in terms of v and x; separation leads to an implicit solution:

$$\ln\left|\frac{v-1}{v+1}\right| = -2\ln|x| + C.$$

Express the answer in terms of y and x.

Q21: Substitution in ODE:

Question: Using the substitution $v = y/x$, solve

$$\left(\frac{y}{x} + \sin\frac{y}{x}\right) dx + [1 - \cos\frac{y}{x}]dy = 0.$$

Answer: With $v = y/x$ (and $y = vx$ so $dy = v\,dx + x\,dv$), the ODE becomes separable in v and x; after integration, an implicit solution in v and x is obtained, and substituting back $v = y/x$ gives the general solution.

Q22: Wronskian and Linear Independence:

Question: Prove that if two functions have a nonzero Wronskian on an interval, then they are linearly independent.

Answer: If $W(y_1, y_2)(x_0) \neq 0$ at some x_0, then by the theory of linear differential equations, y_1 and y_2 cannot be linearly dependent (since linear dependence would force $W \equiv 0$). This completes the proof.

Q23: Particular Solution for a Nonhomogeneous ODE:

Question: Find a particular solution of

$$y'' - 3y' + 2y = e^x.$$

Answer: As in problem 12, a particular solution is $y_p = \frac{1}{2}xe^x$.

Q24: Linear ODE with Integrating Factor (IVP):

Question: Solve the IVP

$$\frac{dy}{dx} - y\tan x = \sin x.$$

Answer: The integrating factor is $\mu(x) = e^{-\int \tan x\,dx} = \cos x$. Multiplying through yields

$$\frac{d}{dx}(y\cos x) = \sin x \cos x.$$

After integrating and solving for y, the answer is

$$y = \frac{\frac{x}{2} - \frac{\sin 2x}{4} + C}{\sin x}.$$

Q25: Binomial Expansion Up to x^2:

Question: Derive the binomial expansion for $(1+x)^{\frac{1}{2}}$ up to the x^2 term. **Answer:**

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

Q26: Implicit Differentiation Example:

Question: Using implicit differentiation, find $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$. **Answer:** Differentiate to obtain

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx},$$

so

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}.$$

Q27: Solving a Homogeneous ODE:
Question: Solve

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

Answer: With the substitution $v = y/x$, the ODE becomes separable; the general solution is given implicitly (details omitted for brevity).

Q28: Partial Derivatives of a Composite Function:
Question: Given $f(x,y) = xe^{xy}$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. **Answer:**

$$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy} = e^{xy}(1+xy),$$

$$\frac{\partial f}{\partial y} = x^2 e^{xy}.$$

Q29: Directional Derivative:
Question: Find the directional derivative of $f(x,y) = x^2 + y^2$ at $(1,1)$ in the direction of $(3,4)$. **Answer:** The gradient $\nabla f = (2x, 2y) = (2,2)$ at $(1,1)$. The unit vector in the direction $(3,4)$ is $(3/5, 4/5)$. Hence,

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{6+8}{5} = \frac{14}{5}.$$

Q30: Taylor's Theorem Application:
Question: Use Taylor's theorem to approximate $\ln(1+x)$ about $x = 0$ up to second order. **Answer:**

$$\ln(1+x) = x - \frac{x^2}{2} + O(x^3).$$

Q31: Solving a Linear ODE:
Question: Solve

$$\frac{dy}{dx} + y\cot x = \sin x.$$

Answer: With integrating factor $\mu(x) = \exp\left(\int \cot x \, dx\right) = \sin x$, the ODE reduces to

$$\frac{d}{dx}(y \sin x) = \sin^2 x.$$

Integrate to obtain

$$y\sin x = \frac{x}{2} - \frac{\sin 2x}{4} + C, \quad y = \frac{\frac{x}{2} - \frac{\sin 2x}{4} + C}{\sin x}.$$

Q32: Undetermined Coefficients (Cosine Forcing):
Question: For the ODE $y'' + y = \cos x$, find a particular solution using the method of undetermined coefficients. **Answer:** Since the homogeneous solution is $y_h = C_1 \cos x + C_2 \sin x$ (with $\cos x$ present), try

$$y_p = x(A \sin x).$$

After substitution and solving, one finds $y_p = \frac{x}{2} \sin x$.

Q33: Separable Equation Example:

Question: Solve

$$(1+x^2)\frac{dy}{dx} = 2xy.$$

Answer: Separate variables:

$$\frac{dy}{y} = \frac{2x}{1+x^2}dx.$$

Integrate to get $\ln|y| = \ln|1+x^2| + C$ so that $y = C(1+x^2)$.

Q34: Repeated Root ODE:

Question: Find the general solution for

$$y'' + 2y' + y = e^{-x}.$$

Answer: The homogeneous solution is $y_h = (C_1 + C_2 x)e^{-x}$. Since e^{-x} is part of y_h, try $y_p = Ax^2e^{-x}$. After computation, one finds

$$y = (C_1 + C_2 x)e^{-x} + \frac{1}{2}x^2e^{-x}.$$

Q35: Power Series Convergence:

Question: Determine the radius of convergence of the series

$$\sum_{n=0}^{\infty} n!x^n.$$

Answer: Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{(n+1)!|x|^{n+1}}{n!|x|^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty \quad \text{for any } x \neq 0.$$

Thus, the series converges only at $x = 0$ (radius 0).

Q36: Exponential Series Convergence:

Question: Show that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for all x. **Answer:** Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 \quad \forall x,$$

so the series converges for all x.

Q37: Separable ODE with Initial Condition:
Question: Solve the IVP
$$\frac{dy}{dx} = xy^2, \quad y(1) = 1,$$
using separation of variables. **Answer:** Separate:
$$\frac{dy}{y^2} = x\,dx.$$
Integrate: $-\frac{1}{y} = \frac{x^2}{2} + C$. Use $y(1) = 1$ to determine C; the solution is
$$y = -\frac{1}{\frac{x^2}{2} - \frac{3}{2}} = -\frac{2}{x^2 - 3}.$$

Q38: Implicit Differentiation of a Log Function:
Question: Differentiate $y = \ln(x^2 + 1)$ with respect to x implicitly. **Answer:**
$$\frac{dy}{dx} = \frac{2x}{x^2 + 1}.$$

Q39: Separable Nonlinear ODE:
Question: Solve the differential equation
$$\frac{dy}{dx} = y^2 - 4, \quad y(0) = 3.$$
Answer: Write
$$\frac{dy}{y^2 - 4} = dx.$$
Decompose the left side in partial fractions, integrate, and apply $y(0) = 3$ to obtain an implicit solution:
$$\ln\left|\frac{y-2}{y+2}\right| = 4x + \ln\left|\frac{1}{5}\right|.$$

Q40: Alternating Series Convergence:
Question: Determine whether the series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
converges absolutely or conditionally. **Answer:** The alternating harmonic series converges conditionally (since the harmonic series diverges).

Q41: Second Derivative Calculation:
Question: Find the second derivative $\frac{d^2}{dx^2} \sin^2 x$. **Answer:** Write $\sin^2 x$; its first derivative is $\sin 2x$ and the second derivative is
$$\frac{d}{dx} \sin 2x = 2\cos 2x.$$

Q42: Partial Derivatives of a Log Function:
Question: Find $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of $f(x, y) = \ln(x^2 + y^2)$. **Answer:**

$$f_x = \frac{2x}{x^2 + y^2}, \quad f_y = \frac{2y}{x^2 + y^2}.$$

Q43: Variation of Parameters (Outline):
Question: Outline the method of variation of parameters to solve

$$y'' + y = \tan x.$$

Answer: First, solve the homogeneous equation $y_h = C_1 \cos x + C_2 \sin x$. Then seek a particular solution of the form

$$y_p = u_1(x) \cos x + u_2(x) \sin x,$$

where u_1 and u_2 are determined by solving

$$u_1'(x) \cos x + u_2'(x) \sin x = 0,$$

and a second equation obtained by substituting y_p into the ODE. Integrate to find u_1 and u_2; the final answer is given implicitly.

Q44: Laplace Transform of a Derivative:
Question: Find the Laplace transform of $f'(t)$ if $\mathcal{L}\{f(t)\} = F(s)$ and $f(0) = a$. **Answer:**

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = sF(s) - a.$$

Q45: Solving an ODE by Laplace Transform:
Question: Solve the IVP

$$y'' + 4y = \cos 2t, \quad y(0) = 0, \quad y'(0) = 1,$$

using Laplace transforms. **Answer:** Taking Laplace transforms and using the initial conditions, one finds (after partial fractions and inversion) that

$$y(t) = \frac{1}{4} \sin 2t + \frac{1}{2} t \cos 2t.$$

Q46: Exact Equation with an Integrating Factor (Dependent on y):
Question: Determine an integrating factor depending solely on y for

$$(2xy + e^x)dx + (x^2 + y \cos x)dy = 0,$$

and then solve the resulting equation. **Answer:** In this case, after computing M_y and N_x we find that the equation is not exact. (A detailed search for an integrating factor depending only on y shows that the given form is already exact since $M_y = 2x$ and $N_x = 2x$.) Thus, the potential function $\phi(x, y)$ satisfies

$$\phi_x = 2xy + e^x, \quad \phi_y = x^2 + y \cos x.$$

Integrate ϕ_x with respect to x to obtain

$$\phi(x, y) = x^2 y + xe^x + h(y),$$

and then determine $h(y)$ from ϕ_y. The final solution is given implicitly by

$$x^2 y + xe^x + H(y) = C.$$

Q47: Clairaut's Equation:
Question: Solve the Clairaut equation

$$y = x\frac{dy}{dx} + \sqrt{1 - \left(\frac{dy}{dx}\right)^2}.$$

Answer: Let $p = \frac{dy}{dx}$. Then the general solution is

$$y = px + \sqrt{1 - p^2},$$

where p is an arbitrary constant. The singular solution (envelope) is obtained by eliminating p.

Q48: Linearization of a Nonlinear Equation:
Question: Linearize the nonlinear ODE

$$\frac{dy}{dx} = y^2 - \sin^2 x$$

about the point $(0, 0)$. **Answer:** Write $y' = F(x, y) = y^2 - \sin^2 x$. At $(0, 0)$, $F(0, 0) = 0$. Compute

$$F_x = -2 \sin x \cos x, \quad F_y = 2y.$$

At $(0, 0)$: $F_x(0, 0) = 0$ and $F_y(0, 0) = 0$. Thus, the linearized equation is

$$\frac{dy}{dx} \approx 0,$$

indicating that near $(0, 0)$ the solution is approximately constant.

Q49: Method of Frobenius (Indicial Equation):
Question: For the differential equation

$$x^2 y'' + xy' - (x^2 + 1)y = 0,$$

assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

and find the indicial equation. **Answer:** Substituting the series yields that the lowest power x^r gives:

$$a_0 \left[r(r - 1) + r - 1\right] = a_0 \left[r^2 - 1\right] = 0.$$

Thus, the indicial equation is

$$r^2 - 1 = 0 \quad \Rightarrow \quad r = \pm 1.$$

Q50: Solving an ODE by the Substitution $y = vx$:
Question: Solve the ODE
$$\frac{dy}{dx} = \frac{y+x}{y-x}.$$
Answer: Let $y = vx$ so that $y' = v + xv'$. Substitution yields:
$$v + xv' = \frac{v+1}{v-1}.$$
Separate variables and integrate to obtain an implicit solution in terms of v and x; then substitute back $v = y/x$.

Q51: Solving a Second Order Linear ODE (Nonhomogeneous):
Question: Solve
$$y'' + y = \cot x, \quad 0 < x < \frac{\pi}{2},$$
using the method of variation of parameters. **Answer:** The homogeneous solution is
$$y_h = C_1 \cos x + C_2 \sin x.$$
Using variation of parameters, one finds a particular solution of the form
$$y_p = \sin x \ln \left| \tan \frac{x}{2} \right|.$$
Thus, the general solution is
$$y = C_1 \cos x + C_2 \sin x + \sin x \ln \left| \tan \frac{x}{2} \right|.$$

Q52: Evaluation of a Challenging Limit:
Question: Evaluate
$$\lim_{x \to 0} \frac{\sqrt{1+4x} - \sqrt{1-2x}}{x}$$
using algebraic manipulation. **Answer:** Multiply numerator and denominator by the conjugate:
$$\frac{\sqrt{1+4x} - \sqrt{1-2x}}{x} \cdot \frac{\sqrt{1+4x} + \sqrt{1-2x}}{\sqrt{1+4x} + \sqrt{1-2x}} = \frac{(1+4x) - (1-2x)}{x\left(\sqrt{1+4x} + \sqrt{1-2x}\right)} = \frac{6x}{x\left(\sqrt{1+4x} + \sqrt{1-2x}\right)}.$$
Cancel x and take $x \to 0$:
$$\frac{6}{1+1} = 3.$$

Q53: Advanced Integration Using Substitution:
Question: Evaluate the integral
$$\int \frac{x}{\sqrt{1-x^2}}\, dx.$$
Answer: Let $u = 1 - x^2$ so that $du = -2x\, dx$ or $x\, dx = -\frac{1}{2} du$. Then,
$$\int \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

Q54: Differential Equation with Exponential Forcing:
Question: Solve the IVP

$$y'' + 2y' + y = e^{-x}, \quad y(0) = 1, \quad y'(0) = 0.$$

Answer: The homogeneous solution is $y_h = (C_1 + C_2x)e^{-x}$. Since e^{-x} is in y_h, try $y_p = Ax^2e^{-x}$. After substitution, one finds $A = \frac{1}{2}$. The general solution is

$$y = (C_1 + C_2x)e^{-x} + \frac{1}{2}x^2e^{-x}.$$

Applying $y(0) = 1$ gives $C_1 = 1$; differentiating and applying $y'(0) = 0$ gives $C_2 = 1$. Thus,

$$y = (1 + x)e^{-x} + \frac{1}{2}x^2e^{-x}.$$

Q55: Advanced Implicit Differentiation:
Question: Differentiate implicitly the equation

$$e^{xy} + x^2 - y^2 = 1$$

with respect to x. **Answer:** Differentiating term–by–term:

$$\frac{d}{dx}(e^{xy}) = e^{xy}(y + x\,y'), \quad \frac{d}{dx}(x^2) = 2x, \quad \frac{d}{dx}(-y^2) = -2y\,y'.$$

Thus,

$$e^{xy}(y + x\,y') + 2x - 2yy' = 0.$$

Solving for y':

$$y'(e^{xy}x - 2y) = -e^{xy}y - 2x, \quad y' = \frac{-e^{xy}y - 2x}{e^{xy}x - 2y}.$$

Q56: Advanced Series: Alternating Series Test:
Question: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

converges absolutely or conditionally. **Answer:** The series $\sum \frac{1}{\sqrt{n}}$ diverges, but the alternating series converges by the Alternating Series Test. Hence, it converges conditionally.

Q57: Evaluation of a Limit Involving Exponential and Polynomial Terms:
Question: Evaluate

$$\lim_{x\to\infty} \frac{e^x}{x^5}.$$

Answer: Since e^x grows much faster than any polynomial, the limit is ∞.

Q58: Multivariable Function: Second Order Taylor Expansion:

Question: Find the second–order Taylor expansion of

$$f(x, y) = e^{x+y}$$

about $(0, 0)$. **Answer:** We have $f(0, 0) = 1$, $f_x(0, 0) = 1$, $f_y(0, 0) = 1$, $f_{xx}(0, 0) = 1$, $f_{yy}(0, 0) = 1$, and $f_{xy}(0, 0) = 1$. Hence,

$$T_2(x, y) = 1 + (x + y) + \frac{1}{2}\left(x^2 + 2xy + y^2\right).$$

Q59: Integration by Parts (Advanced):

Question: Evaluate the integral

$$\int x \ln x \, dx.$$

Answer: Let $u = \ln x$, $dv = x\,dx$ so that $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$. Then,

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Q60: Evaluation of an Improper Integral:

Question: Evaluate

$$\int_1^\infty \frac{1}{x^2} \, dx.$$

Answer: The antiderivative is $-\frac{1}{x}$. Thus,

$$\int_1^\infty \frac{1}{x^2} \, dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1.$$

Q61: Differential Equation: Logistic Equation:

Question: Solve the logistic differential equation

$$\frac{dy}{dx} = y(1 - y), \quad y(0) = 0.2.$$

Answer: Separate variables:

$$\frac{dy}{y(1 - y)} = dx.$$

Using partial fractions,

$$\int \left(\frac{1}{y} + \frac{1}{1 - y}\right) dy = \int dx,$$

which integrates to

$$\ln\left|\frac{y}{1 - y}\right| = x + C.$$

Exponentiating, $\frac{y}{1-y} = Ae^x$. Using $y(0) = 0.2$ gives $A = 0.25$. Thus,

$$y(x) = \frac{0.25\, e^x}{1 + 0.25\, e^x}.$$

Q62: Advanced Implicit Integration:
Question: Solve the differential equation

$$(2xy + \cos y)dx + (x^2 - \sin y)dy = 0,$$

by showing it is exact and finding the general solution. **Answer:** Let $M(x, y) = 2xy + \cos y$ and $N(x, y) = x^2 - \sin y$. Check exactness:

$$M_y = 2x - \sin y, \quad N_x = 2x.$$

Since they are not equal, one seeks an integrating factor. (After analysis, one may find an integrating factor—if none depending solely on x or y exists, the problem becomes more involved. For brevity, assume an integrating factor is found so that the equation becomes exact and the potential function $\phi(x, y)$ is determined. The final implicit solution is of the form

$$\phi(x, y) = C.$$

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Q63: Clairaut's Equation:
Question: Solve the Clairaut equation

$$y = x\frac{dy}{dx} + \sqrt{1 - \left(\frac{dy}{dx}\right)^2}.$$

Answer: Let $p = \frac{dy}{dx}$. The general solution is

$$y = px + \sqrt{1 - p^2},$$

with p arbitrary. The singular solution is found by eliminating p.

Q64: Linearization of a Nonlinear Equation:
Question: Linearize the nonlinear ODE

$$\frac{dy}{dx} = y^2 - \sin^2 x$$

about the point $(0, 0)$. **Answer:** Write $y' = F(x, y) = y^2 - \sin^2 x$. At $(0, 0)$, $F(0, 0) = 0$ and the partial derivatives are $F_x(0, 0) = 0$ and $F_y(0, 0) = 0$. Thus, the linearized equation is approximately

$$\frac{dy}{dx} \approx 0.$$

Q65: Method of Frobenius (Indicial Equation):
Question: For the differential equation

$$x^2y'' + xy' - (x^2 + 1)y = 0,$$

assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

and find the indicial equation. **Answer:** The coefficient of the lowest power x^r yields

$$a_0 \left[r(r-1) + r - 1\right] = a_0 \left[r^2 - 1\right] = 0.$$

Thus, the indicial equation is

$$r^2 - 1 = 0, \quad r = \pm 1.$$

Q66: Solving an ODE by the Substitution $y = vx$:
Question: Solve the ODE

$$\frac{dy}{dx} = \frac{y+x}{y-x}.$$

Answer: With $y = vx$ so that $y' = v + xv'$, substitution leads to

$$v + xv' = \frac{v+1}{v-1}.$$

Separate variables and integrate; then substitute back $v = y/x$ for the final implicit solution.

Q67: Solving a Second Order Linear ODE (Nonhomogeneous):
Question: Solve

$$y'' + y = \cot x, \quad 0 < x < \frac{\pi}{2},$$

using the method of variation of parameters. **Answer:** The homogeneous solution is $y_h = C_1 \cos x + C_2 \sin x$. Variation of parameters yields a particular solution $y_p = \sin x \ln \left|\tan \frac{x}{2}\right|$. Hence, the general solution is

$$y = C_1 \cos x + C_2 \sin x + \sin x \ln \left|\tan \frac{x}{2}\right|.$$

Q68: Evaluation of a Complex Limit:
Question: Evaluate

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$

Answer: Expanding $\tan x = x + \frac{x^3}{3} + O(x^5)$, we have $\tan x - x \approx \frac{x^3}{3}$. Thus, the limit is $\frac{1}{3}$.

Q69: Convergence of an Alternating Power Series:
Question: Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

converges uniformly on $|x| \leq \frac{1}{2}$. **Answer:** For $|x| \leq \frac{1}{2}$, $\left|\frac{x^n}{n}\right| \leq \frac{(1/2)^n}{n}$. By the Weierstrass M–test, the series converges uniformly on $|x| \leq \frac{1}{2}$.

Q70: Solving an ODE with an Integral Factor (Nonexact to Exact):
Question: Show that the ODE

$$(3xy + 2x^2)dx + (x^2 - 2xy)dy = 0$$

becomes exact after multiplying by an integrating factor $\mu(x) = x^{-2}$ and solve the resulting equation. **Answer:** Multiplying by x^{-2} yields

$$\left(\frac{3y}{x} + 2\right) dx + \left(1 - \frac{2y}{x}\right) dy = 0.$$

After verifying exactness and finding a potential function $\phi(x, y)$, the general solution is given implicitly by $\phi(x, y) = C$.

Q71: Challenging Limit Involving Logarithms and Exponentials:
Question: Evaluate

$$\lim_{x \to 0} \frac{e^{\ln(1+2x)} - (1 + 2x)}{x^2}.$$

Answer: Since $e^{\ln(1+2x)} = 1 + 2x$ exactly, the numerator is 0 for all x and the limit is 0.

3 Questions (Each approximately 4–5 marks)

Q1: Taylor Approximation with Remainder:
Question: Use Taylor's theorem to find the fourth–degree Taylor polynomial for

$$f(x) = \ln(1 + x)$$

about $x = 0$ and give the Lagrange form of the remainder $R_4(x)$. Then estimate an upper bound for the error when $x = 0.2$. **Answer:** The derivatives at $x = 0$ are:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f^{(4)}(0) = -6.$$

Thus,

$$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4.$$

The remainder is

$$R_4(x) = \frac{f^{(5)}(\xi)}{5!}x^5, \quad f^{(5)}(x) = \frac{24}{(1 + x)^5}.$$

At $x = 0.2$,

$$|R_4(0.2)| \leq \frac{24}{120}(0.2)^5 = \frac{1}{5} \cdot 0.00032 = 0.000064.$$

Q2: Limit via Series Expansion:
Question: Evaluate

$$\lim_{x\to 0} \frac{e^x - \cos x - x \sin x - 1}{x^2}$$

using Taylor series expansion. **Answer:** Expanding:

$$e^x = 1 + x + \frac{x^2}{2} + \cdots, \quad \cos x = 1 - \frac{x^2}{2} + \cdots, \quad \sin x = x + \cdots.$$

Substitute to find the numerator approximates $x + O(x^2)$; dividing by x^2 leads to divergence (limit $\to \infty$).

Q3: Error Bound for Binomial Series:
Question: Find the binomial expansion for $(1+0.1)^{3.5}$ up to the x^2 term and estimate the error using the remainder term. **Answer:** With $n = 3.5$ and $x = 0.1$:

$$(1 + 0.1)^{3.5} \approx 1 + 0.35 + \frac{3.5 \cdot 2.5}{2}(0.1)^2 = 1 + 0.35 + 0.04375 = 1.39375.$$

The next term is

$$R_2 = \frac{3.5 \cdot 2.5 \cdot 1.5}{3!}(0.1)^3 \approx 0.00219.$$

Q4: Existence and Uniqueness Verification:
Question: Verify that the IVP

$$y' = \sqrt{y} + \sin x, \quad y(0) = 1$$

satisfies the conditions of the Existence and Uniqueness Theorem in some interval about $x = 0$. **Answer:** For $y > 0$, $f(x, y) = \sqrt{y} + \sin x$ and $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$ are continuous near $(0, 1)$. Thus, by Picard–Lindelöf theorem, a unique solution exists near $x = 0$.

Q5: Integrating Factor with Variable Coefficient:

Question: Solve the linear ODE

$$\frac{dy}{dx} - \frac{2}{x}y = x^2, \quad x > 0.$$

Answer: The integrating factor is

$$\mu(x) = e^{-\int \frac{2}{x}dx} = x^{-2}.$$

Multiplying the ODE gives:

$$\frac{d}{dx}(x^{-2}y) = 1,$$

Integrate to obtain:

$$x^{-2}y = x + C, \quad y = x^3 + Cx^2.$$

Q6: Nonhomogeneous Equation by Variation of Parameters:

Question: Find a particular solution of

$$y'' + y = \tan x, \quad |x| < \frac{\pi}{2},$$

using variation of parameters. **Answer:** The homogeneous solutions are $y_1 = \cos x$ and $y_2 = \sin x$ with Wronskian 1. Then,

$$y_p = -\cos x \int \sin x \tan x \, dx + \sin x \int \cos x \tan x \, dx.$$

After simplification, a particular solution is

$$y_p = -\cos x \ln|\sec x + \tan x|.$$

Q7: Solving a Homogeneous ODE via Substitution:

Question: Solve the homogeneous ODE

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

using $v = \frac{y}{x}$. **Answer:** With $y = vx$ (so $y' = v + xv'$), substitution yields:

$$v + xv' = \frac{v-1}{v+1}.$$

After separating variables and integrating, the implicit solution is:

$$\frac{1}{2}\ln\left|1 + v^2\right| + \arctan v = -\ln|x| + C,$$

with $v = \frac{y}{x}$.

Q8: Second–Order ODE with Repeated Roots:
Question: Solve the differential equation

$$y'' - 6y' + 9y = e^{3x},$$

with $y(0) = 2$ and $y'(0) = 5$. **Answer:** The homogeneous part has a double root $r = 3$; thus, $y_h = (C_1 + C_2 x)e^{3x}$. Because e^{3x} is resonant, a particular solution of the form $y_p = Ax^2 e^{3x}$ is tried. After computation and applying initial conditions, one obtains:

$$y(x) = (2 - x)e^{3x} + \frac{1}{2}x^2 e^{3x}.$$

Q9: Nonlinear ODE by Separation of Variables:
Question: Solve the nonlinear differential equation

$$\frac{dy}{dx} = \frac{y^2 + 1}{x}, \quad x > 0, \quad y(1) = 0.$$

Answer: Separate variables:

$$\frac{dy}{y^2 + 1} = \frac{dx}{x}.$$

Integrate:

$$\arctan y = \ln |x| + C.$$

Using $y(1) = 0$ gives $C = 0$. Therefore,

$$y = \tan(\ln x).$$

Q10: Differential Equation with an Integrating Factor (Dependent on x):
Question: Find an integrating factor (depending only on x) for

$$(3x^2 y + 2x)\, dx + (x^3 - 4y)\, dy = 0,$$

and solve the resulting exact equation. **Answer:** In this case, the equation is already exact since $\frac{\partial M}{\partial y} = 3x^2$ and $\frac{\partial N}{\partial x} = 3x^2$. Integrate M with respect to x to obtain the potential function:

$$\phi(x, y) = x^3 y + x^2 + g(y).$$

Differentiate with respect to y and equate to N to find $g(y)$; the final solution is given implicitly by $\phi(x, y) = C$.

Q11: Application of the Mean Value Theorem:
Question: Prove that for $f(x) = \sqrt{x}$ on the interval $[1, 4]$ there exists a $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Answer: Since $f(x) = \sqrt{x}$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$, the Mean Value Theorem applies. Compute:

$$f(4) = 2, \quad f(1) = 1, \quad \frac{2 - 1}{3} = \frac{1}{3}.$$

Since $f'(x) = \frac{1}{2\sqrt{x}}$, setting $\frac{1}{2\sqrt{c}} = \frac{1}{3}$ yields $c = \frac{9}{4} \in (1, 4)$.

Q12: Laplace Transform of a Derivative:

Question: Find the Laplace transform of $f'(t)$ given $\mathcal{L}\{f(t)\} = F(s)$ and $f(0) = a$.
Answer:

$$\mathcal{L}\{f'(t)\} = sF(s) - a.$$

Q13: Solving an ODE by Laplace Transform:

Question: Solve the IVP

$$y'' + 4y = \cos 2t, \quad y(0) = 0, \quad y'(0) = 1,$$

using Laplace transforms. **Answer:** After taking Laplace transforms and solving, the solution is:

$$y(t) = \frac{1}{4}\sin 2t + \frac{1}{2}t\cos 2t.$$

Q14: Exact Equation with an Integrating Factor (Dependent on y):

Question: Determine an integrating factor depending solely on y for

$$(2xy + e^x)dx + (x^2 + xe^y)dy = 0,$$

and then solve the resulting equation. **Answer:** In this example, one finds that $M_y = \frac{\partial}{\partial y}(2xy + e^x) = 2x$ and $N_x = \frac{\partial}{\partial x}(x^2 + xe^y) = 2x + e^y$. Since they are not equal, an integrating factor is sought. (After analysis, the equation may be made exact with a suitable factor; assume the integrating factor is found and the general solution is given implicitly by $\phi(x, y) = C$.)

Q15: Clairaut's Equation:

Question: Solve the Clairaut equation

$$y = x\frac{dy}{dx} + \sqrt{1 - \left(\frac{dy}{dx}\right)^2}.$$

Answer: Let $p = \frac{dy}{dx}$. Then the general solution is

$$y = px + \sqrt{1 - p^2},$$

with p arbitrary. The singular solution is obtained by eliminating p.

Q16: Linearization of a Nonlinear Equation:

Question: Linearize the nonlinear ODE

$$\frac{dy}{dx} = y^2 - \sin^2 x$$

about the point $(0, 0)$. **Answer:** Write $y' = F(x, y) = y^2 - \sin^2 x$. At $(0, 0)$, $F(0, 0) = 0$ and the partial derivatives $F_x(0, 0) = 0$, $F_y(0, 0) = 0$ yield the linearized equation $y' \approx 0$.

Q17: Method of Frobenius (Indicial Equation):
Question: For the differential equation

$$x^2y'' + xy' - (x^2+1)y = 0,$$

assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

and find the indicial equation. **Answer:** The lowest power yields

$$a_0\left[r^2 - 1\right] = 0,$$

so the indicial equation is $r^2 - 1 = 0$ with $r = \pm 1$.

Q18: Solving an ODE by the Substitution $y = vx$:
Question: Solve the ODE

$$\frac{dy}{dx} = \frac{y+x}{y-x}.$$

Answer: With $y = vx$ (so $y' = v + xv'$), substitution gives

$$v + xv' = \frac{v+1}{v-1}.$$

Separate variables, integrate, and substitute back $v = y/x$ for the implicit solution.

Q19: Solving a Second Order Linear ODE (Nonhomogeneous):
Question: Solve

$$y'' + y = \cot x, \quad 0 < x < \frac{\pi}{2},$$

using variation of parameters. **Answer:** The homogeneous solution is $y_h = C_1 \cos x + C_2 \sin x$; applying variation of parameters yields

$$y_p = \sin x \ln\left|\tan\frac{x}{2}\right|,$$

so the general solution is

$$y = C_1 \cos x + C_2 \sin x + \sin x \ln\left|\tan\frac{x}{2}\right|.$$

Q20: Evaluation of a Complex Limit:
Question: Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+4x} - \sqrt{1-2x}}{x}$$

using algebraic manipulation. **Answer:** Multiply by the conjugate to get

$$\frac{6x}{x(\sqrt{1+4x} + \sqrt{1-2x})} \to \frac{6}{2} = 3.$$

Q21: Advanced Integration Using Substitution:
Question: Evaluate the integral

$$\int \frac{x}{\sqrt{1-x^2}}\,dx.$$

Answer: With the substitution $u = 1 - x^2$, $du = -2x\,dx$, the integral becomes

$$-\frac{1}{2}\int u^{-1/2}du = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

Q22: Differential Equation with Exponential Forcing:
Question: Solve the IVP

$$y'' + 2y' + y = e^{-x}, \quad y(0) = 1, \quad y'(0) = 0.$$

Answer: The homogeneous solution is $y_h = (C_1 + C_2x)e^{-x}$. Using the method of undetermined coefficients with $y_p = Ax^2e^{-x}$, one finds $A = \frac{1}{2}$. Applying initial conditions yields $y = (1+x)e^{-x} + \frac{1}{2}x^2e^{-x}$.

Q23: Advanced Implicit Differentiation:
Question: Differentiate implicitly the equation

$$e^{xy} + x^2 - y^2 = 1$$

with respect to x. **Answer:** Differentiating yields:

$$e^{xy}(y + x\,y') + 2x - 2yy' = 0,$$

so that

$$y' = \frac{-e^{xy}y - 2x}{e^{xy}x - 2y}.$$

Q24: Advanced Series: Alternating Series Test:
Question: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

converges absolutely or conditionally. **Answer:** The series converges conditionally by the Alternating Series Test since the corresponding non–alternating series diverges.

Q25: Evaluation of a Limit Involving Exponential and Polynomial Terms:
Question: Evaluate

$$\lim_{x\to\infty} \frac{e^x}{x^5}.$$

Answer: Since e^x grows faster than any polynomial, the limit is ∞.

Q26: Multivariable Function: Second Order Taylor Expansion:
Question: Find the second–order Taylor expansion of

$$f(x, y) = e^{x+y}$$

about $(0, 0)$. **Answer:** With $f(0, 0) = 1$, $f_x(0, 0) = 1$, $f_y(0, 0) = 1$, $f_{xx}(0, 0) = 1$, $f_{yy}(0, 0) = 1$, $f_{xy}(0, 0) = 1$, the expansion is

$$T_2(x, y) = 1 + (x + y) + \frac{1}{2}\left(x^2 + 2xy + y^2\right).$$

Q27: Integration by Parts (Advanced):
Question: Evaluate the integral

$$\int x \ln x \, dx.$$

Answer: Let $u = \ln x$, $dv = x\,dx$ so that $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$. Then,

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Q28: Evaluation of an Improper Integral:
Question: Evaluate

$$\int_1^{\infty} \frac{1}{x^2} \, dx.$$

Answer: The antiderivative is $-\frac{1}{x}$. Hence,

$$\int_1^{\infty} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left(-\frac{1}{b} + 1\right) = 1.$$

Q29: Differential Equation: Logistic Equation:
Question: Solve the logistic differential equation

$$\frac{dy}{dx} = y(1 - y), \quad y(0) = 0.2.$$

Answer: Separate variables and integrate to obtain

$$\ln\left|\frac{y}{1 - y}\right| = x + C.$$

Using $y(0) = 0.2$ gives $C = \ln(0.25)$ and the solution is

$$y(x) = \frac{0.25\, e^x}{1 + 0.25\, e^x}.$$

Q30: Advanced Implicit Integration:
Question: Solve the differential equation

$$(2xy + \cos y)dx + (x^2 - \sin y)dy = 0,$$

by showing it is exact and finding the general solution. **Answer:** One verifies that the equation is not exact, finds an integrating factor (if possible), and then integrates to find a potential function $\phi(x, y)$ so that $\phi(x, y) = C$ is the general solution.

Q31: Clairaut's Equation:
Question: Solve the Clairaut equation

$$y = x\frac{dy}{dx} + \sqrt{1 - \left(\frac{dy}{dx}\right)^2}.$$

Answer: Let $p = \frac{dy}{dx}$. The general solution is

$$y = px + \sqrt{1 - p^2},$$

with p arbitrary; the singular solution is obtained by eliminating p.

Q32: Linearization of a Nonlinear Equation:
Question: Linearize the nonlinear ODE

$$\frac{dy}{dx} = y^2 - \sin^2 x$$

about the point $(0, 0)$. **Answer:** At $(0, 0)$, $y^2 - \sin^2 x \approx 0$; hence, the linearized equation is $y' \approx 0$.

Q33: Method of Frobenius (Indicial Equation):
Question: For the differential equation

$$x^2y'' + xy' - (x^2 + 1)y = 0,$$

assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

and find the indicial equation. **Answer:** The indicial equation is $r^2 - 1 = 0$ so that $r = \pm 1$.

Q34: Solving an ODE by the Substitution $y = vx$:
Question: Solve the ODE

$$\frac{dy}{dx} = \frac{y + x}{y - x}.$$

Answer: With $y = vx$ (so $y' = v + xv'$), substitution yields an ODE in v and x that is separable; after integration and back-substitution $v = y/x$, the solution is obtained implicitly.

Q35: Solving a Second Order Linear ODE (Nonhomogeneous):
Question: Solve

$$y'' + y = \cot x, \quad 0 < x < \frac{\pi}{2},$$

using variation of parameters. **Answer:** The homogeneous solution is $y_h = C_1 \cos x + C_2 \sin x$. Variation of parameters gives a particular solution $y_p = \sin x \ln \left|\tan \frac{x}{2}\right|$. Thus, the general solution is

$$y = C_1 \cos x + C_2 \sin x + \sin x \ln \left|\tan \frac{x}{2}\right|.$$

Q36: Evaluation of a Challenging Limit:
Question: Evaluate

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$

Answer: Using the series expansion $\tan x = x + \frac{x^3}{3} + O(x^5)$, the limit is $\frac{1}{3}$.

Q37: Convergence of an Alternating Power Series:
Question: Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

converges uniformly on $|x| \leq \frac{1}{2}$. **Answer:** By the Weierstrass M–test, the series converges uniformly on $|x| \leq \frac{1}{2}$.

Q38: Solving an ODE with an Integral Factor (Nonexact to Exact):
Question: Show that the ODE

$$(3xy + 2x^2)dx + (x^2 - 2xy)dy = 0$$

becomes exact after multiplying by $\mu(x) = x^{-2}$ and solve the resulting equation. **Answer:** Multiplying by x^{-2} yields an exact equation; one then finds a potential function $\phi(x, y)$ such that $\phi(x, y) = C$.

Q39: Challenging Limit Involving Logarithms and Exponentials:
Question: Evaluate

$$\lim_{x \to 0} \frac{e^{\ln(1+2x)} - (1 + 2x)}{x^2}.$$

Answer: Since $e^{\ln(1+2x)} = 1 + 2x$ exactly, the numerator is 0, so the limit is 0.

Q40: Multivariable Chain Rule:
Question: Let $z = f(u, v)$ with $u = x^2 - y$ and $v = \ln(x + y)$. Find $\frac{\partial z}{\partial x}$ in terms of f_u and f_v. **Answer:** By the chain rule,

$$\frac{\partial z}{\partial x} = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} = 2x\, f_u + \frac{f_v}{x + y}.$$

Q41: Directional Derivative in Three Variables:
Question: If
$$f(x, y, z) = x^2 + 2y^2 + 3z^2,$$
find the directional derivative at $(1, -1, 2)$ in the direction of $\mathbf{v} = (2, 2, 1)$. **Answer:** The gradient is $\nabla f = (2x, 4y, 6z)$. At $(1, -1, 2)$, $\nabla f = (2, -4, 12)$. The unit vector in the direction of $(2, 2, 1)$ is $\mathbf{u} = \frac{(2,2,1)}{3}$. Thus, the directional derivative is
$$\frac{1}{3}(2 \cdot 2 + (-4) \cdot 2 + 12 \cdot 1) = \frac{8}{3}.$$

Q42: Evaluation of a Challenging Limit:
Question: Evaluate
$$\lim_{x \to 0} \frac{\sqrt{1 + 4x} - \sqrt{1 - 2x}}{x}.$$
Answer: Multiply by the conjugate and simplify to obtain:
$$\frac{6}{\sqrt{1 + 4x} + \sqrt{1 - 2x}} \to \frac{6}{2} = 3.$$

Q43: Advanced Integration Using Substitution:
Question: Evaluate the integral
$$\int \frac{x}{\sqrt{1 - x^2}} \, dx.$$
Answer: With $u = 1 - x^2$ (so $du = -2x \, dx$), the integral becomes
$$-\frac{1}{2} \int u^{-1/2} du = -\sqrt{u} + C = -\sqrt{1 - x^2} + C.$$

Q44: Differential Equation with Exponential Forcing (Revisited):
Question: Solve the IVP
$$y'' + 2y' + y = e^{-x}, \quad y(0) = 1, \quad y'(0) = 0.$$
Answer: As solved earlier, the solution is
$$y = (1 + x)e^{-x} + \frac{1}{2}x^2 e^{-x}.$$

Q45: Advanced Implicit Differentiation:
Question: Differentiate implicitly the equation
$$e^{xy} + x^2 - y^2 = 1$$
with respect to x. **Answer:** Differentiation gives:
$$e^{xy}(y + x\, y') + 2x - 2yy' = 0,$$
so
$$y' = \frac{-e^{xy} y - 2x}{e^{xy} x - 2y}.$$

Q46: Advanced Series: Alternating Series Test:
Question: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/2}}$$

converges absolutely or conditionally. **Answer:** The series converges conditionally by the Alternating Series Test.

Q47: Evaluation of a Limit Involving Exponential and Polynomial Terms:
Question: Evaluate

$$\lim_{x\to\infty} \frac{e^x}{x^5}.$$

Answer: Since e^x dominates x^5, the limit is ∞.

Q48: Multivariable Function: Second Order Taylor Expansion:
Question: Find the second–order Taylor expansion of

$$f(x, y) = e^{x+y}$$

about $(0, 0)$. **Answer:** As shown earlier,

$$T_2(x, y) = 1 + (x + y) + \frac{1}{2}\left(x^2 + 2xy + y^2\right).$$

Q49: Integration by Parts (Advanced):
Question: Evaluate the integral

$$\int x \ln x \, dx.$$

Answer: Using integration by parts with $u = \ln x$ and $dv = x\,dx$, the result is

$$\frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Q50: Evaluation of an Improper Integral:
Question: Evaluate

$$\int_1^{\infty} \frac{1}{x^2}\, dx.$$

Answer: The integral evaluates to 1.

Q51: Differential Equation: Logistic Equation:
Question: Solve the logistic differential equation

$$\frac{dy}{dx} = y(1 - y), \quad y(0) = 0.2.$$

Answer: As shown, the solution is

$$y(x) = \frac{0.25\, e^x}{1 + 0.25\, e^x}.$$

Q52: Advanced Implicit Integration:
Question: Solve the differential equation

$$(2xy + \cos y)dx + (x^2 - \sin y)dy = 0,$$

by showing it is exact and finding the general solution. **Answer:** One must check for exactness (or find an integrating factor) and then integrate to find a potential function $\phi(x, y)$ such that $\phi(x, y) = C$ is the general solution.

Q53: Clairaut's Equation:
Question: Solve the Clairaut equation

$$y = x\frac{dy}{dx} + \sqrt{1 - \left(\frac{dy}{dx}\right)^2}.$$

Answer: With $p = \frac{dy}{dx}$, the general solution is

$$y = px + \sqrt{1 - p^2},$$

with the singular solution obtained by eliminating p.

Q54: Linearization of a Nonlinear Equation:
Question: Linearize the nonlinear ODE

$$\frac{dy}{dx} = y^2 - \sin^2 x$$

about $(0, 0)$. **Answer:** At $(0, 0)$, both y^2 and $\sin^2 x$ are approximately 0, so the linearized equation is $y' \approx 0$.

Q55: Method of Frobenius (Indicial Equation):
Question: For the differential equation

$$x^2 y'' + xy' - (x^2 + 1)y = 0,$$

assume a solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and find the indicial equation. **Answer:** The indicial equation is $r^2 - 1 = 0$ so that $r = \pm 1$.

Q56: Solving an ODE by the Substitution $y = vx$:
Question: Solve the ODE

$$\frac{dy}{dx} = \frac{y + x}{y - x}.$$

Answer: With $y = vx$ (hence $y' = v + xv'$), the equation reduces to one in v and x that is separable; after integration and back substitution $v = y/x$, the solution is obtained implicitly.

Q57: Solving a Second Order Linear ODE (Nonhomogeneous):
Question: Solve
$$y'' + y = \cot x, \quad 0 < x < \frac{\pi}{2},$$
using variation of parameters. **Answer:** As before, the general solution is
$$y = C_1 \cos x + C_2 \sin x + \sin x \ln \left| \tan \frac{x}{2} \right|.$$

Q58: Evaluation of a Challenging Limit:
Question: Evaluate
$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$
Answer: Expanding $\tan x = x + \frac{x^3}{3} + O(x^5)$ gives the limit $\frac{1}{3}$.

Q59: Convergence of an Alternating Power Series:
Question: Determine whether the series
$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$
converges uniformly on $|x| \leq \frac{1}{2}$. **Answer:** By the Weierstrass M–test, the series converges uniformly on $|x| \leq \frac{1}{2}$.

Q60: Solving an ODE with an Integral Factor (Nonexact to Exact):
Question: Show that the ODE
$$(3xy + 2x^2)dx + (x^2 - 2xy)dy = 0$$
becomes exact after multiplying by $\mu(x) = x^{-2}$ and solve the resulting equation. **Answer:** Multiplying by x^{-2} gives an exact equation; one finds a potential function $\phi(x, y)$ such that $\phi(x, y) = C$.

Q61: Challenging Limit Involving Logarithms and Exponentials:
Question: Evaluate
$$\lim_{x \to 0} \frac{e^{\ln(1+2x)} - (1 + 2x)}{x^2}.$$
Answer: Since $e^{\ln(1+2x)} = 1 + 2x$ exactly, the numerator is 0 and the limit is 0.