Semester I
DS-1
Mathematical Methods-I (Theory)
PHSDSC101T
(Calculus)
LEC-1-12

Study Material

Lecture 1

1 Limits

Definition: Limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x gets closer to a.

$$\lim_{x \to a} f(x) = L$$

Left-hand limit (LHL) and **Right-hand limit (RHL)**:

$$\lim_{x \to a^-} f(x), \quad \lim_{x \to a^+} f(x)$$

Limit exists only if LHL = RHL.

Important Limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0}(1+x)^{1/x} = e, \quad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

2 Continuity

Definition: A function $f(x)$ is continuous at $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

Key Conditions:

- $f(a)$ is defined.
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x) = f(a)$.

Types of Discontinuity:

- Jump Discontinuity
- Infinite Discontinuity
- Removable Discontinuity

3 Average and Instantaneous Quantities

Average Rate of Change over $[a, b]$:

$$\frac{f(b) - f(a)}{b - a}$$

Instantaneous Rate of Change at $x = a$:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is the **derivative** $f'(a)$.

Physical Interpretation: In motion, Average velocity vs. Instantaneous velocity.

Lecture 2

4 Differentiation

Definition: The derivative $f'(x)$ gives the **rate of change** of f with respect to x.

Notation: $f'(x)$, $\frac{dy}{dx}$, $D_x f(x)$

Basic Derivatives:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Rules of Differentiation:

- Sum Rule: $(f + g)' = f' + g'$
- Product Rule: $(fg)' = f'g + fg'$
- Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- Chain Rule: If $y = f(g(x))$, then
$$\frac{dy}{dx} = \frac{dy}{dg} \times \frac{dg}{dx}$$

5 Plotting Functions

Key Steps:

1. **Domain**: Where is the function defined?
2. **Intercepts**:

- x-intercept: set $f(x) = 0$
- y-intercept: set $x = 0$

3. **Symmetry**:
   - Even function: symmetric about y-axis.
   - Odd function: symmetric about the origin.
4. **Asymptotes**: Vertical, horizontal, or oblique.
5. **Critical Points**: $f'(x) = 0$ for maxima/minima.
6. **Increasing/Decreasing**: Use sign of $f'(x)$.

Examples to Sketch:

- $y = x$ (Line)
- $y = x^2$ (Parabola)
- $y = \sin x$ (Wave)
- $y = e^x$ (Exponential curve)

Summary Table

Concept	**Key Idea**
Limit	Behavior near a point
Continuity	No jumps, breaks, or holes
Average Rate	Change over an interval
Instantaneous Rate	Derivative at a point
Differentiation	Slope, rate of change
Plotting	Understand graph shape

6 Intuitive Idea of Continuous Functions

Continuity: A function is continuous if its graph can be drawn without lifting the pen from the paper.

Formal Understanding: A function $f(x)$ is continuous at $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

Examples:

- Polynomial functions are continuous everywhere.
- Functions like $\frac{1}{x}$ are not continuous at $x = 0$.

Graphical View: No breaks, jumps, or holes.

Lecture 3

7 Intuitive Idea of Differentiable Functions

Differentiability: A function is differentiable at a point if it has a well-defined tangent (non-vertical) at that point.

Formal Understanding: $f(x)$ is differentiable at $x = a$ if:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$

Important Notes:

- Every differentiable function is continuous, but the converse is not true.
- $|x|$ is continuous everywhere but not differentiable at $x = 0$.

Graphical View: Sharp corners or cusps (like $|x|$) are not differentiable.

8 Plotting Curves

Step-by-Step Method:

1. **Find Domain**: Identify where the function is defined.
2. **Intercepts**:
   - x-intercept: solve $f(x) = 0$.
   - y-intercept: evaluate $f(0)$.
3. **Symmetry**:
   - Even function: symmetric about the y-axis.
   - Odd function: symmetric about the origin.
4. **Asymptotes**: Find vertical, horizontal, or oblique asymptotes.
5. **Critical Points**: Find points where $f'(x) = 0$ (maxima, minima).
6. **Concavity and Inflection Points**: Analyze $f''(x)$.

Examples of Curves:

- $y = x^2$: Parabola opening upwards.
- $y = \sin x$: Wave with amplitude 1.
- $y = e^x$: Exponential growth.
- $y = \ln x$: Logarithmic curve (only defined for $x > 0$).

Summary

- Continuity → no breaks.
- Differentiability → smooth curve, no sharp points.
- Good plotting requires domain analysis, critical points, and asymptotic behavior.

Lecture 4

9 Taylor Series (Statement Only)

A function $f(x)$ can be approximated near $x = a$ using its derivatives:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This is called the **Taylor series** expansion around $x = a$.

Special Case (Maclaurin Series): When $a = 0$,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Common Examples:

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

10 Binomial Series (Statement Only)

The binomial series for $(1 + x)^n$, where n is any real number, is given by:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

valid for $|x| < 1$.

11 Convergence of Taylor Series

Convergence Condition: The Taylor series of $f(x)$ converges to $f(x)$ within a certain interval around a if the remainder $R_n(x)$ tends to zero as $n \to \infty$.

Radius of Convergence: The series converges if $|x - a| < R$, where R is the radius of convergence.

12 Tests for Convergence

Ratio Test: Given a series $\sum a_n$, the ratio test states:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

- If $L < 1$, the series converges absolutely.
- If $L > 1$ or $L = \infty$, the series diverges.
- If $L = 1$, the test is inconclusive.

Root Test: Given a series $\sum a_n$, the root test states:

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

- If $L < 1$, the series converges absolutely.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Summary

- Taylor and binomial series are powerful tools for approximating functions.
- Convergence of Taylor series depends on the distance from the center a.
- Ratio and Root Tests help determine convergence.

Lecture 5

13 First Order Differential Equations

A **first-order differential equation** involves the first derivative of a function but no higher derivatives.

General form:

$$\frac{dy}{dx} = f(x, y)$$

Types:

- Separable Equations: $\frac{dy}{dx} = g(x)h(y)$
- Linear Equations: $\frac{dy}{dx} + P(x)y = Q(x)$

14 Integrating Factor (I.F.) Method

For linear first-order differential equations $\frac{dy}{dx} + P(x)y = Q(x)$:

Steps:

1. Find the integrating factor (I.F.):
$$\text{I.F.} = e^{\int P(x)\,dx}$$
2. Multiply through the equation by I.F.
3. Recognize the left-hand side as the derivative of (I.F. $\times\, y$).

4. Integrate both sides to solve for y.

Example: Solve $\frac{dy}{dx} + 2y = e^{-2x}$

- I.F. $= e^{\int 2dx} = e^{2x}$
- Multiply through: $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 1$
- Left-hand side becomes $\frac{d}{dx}(e^{2x}y)$
- Integrate both sides and solve.

15 Second Order Differential Equations

A **second-order differential equation** involves the second derivative $\frac{d^2y}{dx^2}$.

General form:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

Special Case - Homogeneous Equation:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Lecture 6

16 Examples of Second Order Equations

- Simple Harmonic Motion: $\frac{d^2y}{dt^2} + \omega^2 y = 0$
- Damped Motion: $\frac{d^2y}{dt^2} + 2\beta\frac{dy}{dt} + \omega^2 y = 0$

17 Solving Techniques (Overview)

- Finding characteristic equations.
- Using auxiliary equations.
- General solution = Complementary function + Particular integral.

Summary

- First-order differential equations often solved using integrating factors.
- Second-order differential equations arise in physical systems like oscillations.
- Solutions involve systematic approaches based on the type of equation.

18 Homogeneous Second Order Differential Equations with Constant Coefficients

Consider the homogeneous second-order differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where a, b, c are constants.

Solution Method:

1. Form the auxiliary (characteristic) equation:

$$ar^2 + br + c = 0$$

2. Find roots r.
3. General solution depends on the nature of the roots:
   - **Distinct real roots** r_1, r_2:
   $$y = C_1e^{r_1x} + C_2e^{r_2x}$$
   - **Repeated real roots** r:
   $$y = (C_1 + C_2x)e^{rx}$$
   - **Complex conjugate roots** $\alpha \pm i\beta$:
   $$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

Lecture 7

19 Inhomogeneous Second Order Differential Equations with Constant Coefficients

Consider the inhomogeneous second-order differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = F(x)$$

where $F(x)$ is a nonzero function.

Solution Method:

1. Solve the homogeneous equation to find the **complementary function (CF)**.
2. Find a **particular solution (PS)** corresponding to $F(x)$.
3. General solution:
$$y = \mathrm{CF} + \mathrm{PS}$$

Methods to find Particular Solution:

- Method of undetermined coefficients.
- Variation of parameters.

Summary

- Homogeneous second-order differential equations are solved using characteristic equations.
- Inhomogeneous second-order equations require finding both CF and PS.

20 Particular Integral and Wronskian

Particular Integral (PI): The specific solution corresponding to the non-homogeneous part $F(x)$.

Wronskian: Given two functions $y_1(x)$ and $y_2(x)$, the Wronskian is defined as:

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

Use of Wronskian:

- If $W(y_1, y_2)(x) \neq 0$ for all x in an interval, y_1 and y_2 are linearly independent.
- Linearly independent solutions form the basis of the general solution.

Lecture 8

21 Existence and Uniqueness Theorem

Statement: Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous in some rectangle containing (x_0, y_0), then there exists a unique solution $y(x)$ passing through (x_0, y_0).

22 Summary

- Particular Integral gives the effect of nonhomogeneous terms.
- Wronskian helps determine independence of solutions.
- Existence and Uniqueness Theorem ensures that solutions exist and are unique under certain conditions.

23 Functions of Several Variables

A function of two variables assigns to each ordered pair (x, y) in a domain D a unique real number $f(x, y)$.

Example:

$$f(x, y) = x^2 + y^2$$

Similarly, for three variables:

$$f(x, y, z) = x^2 + y^2 + z^2$$

24 Concept of Partial Derivatives

Partial derivatives measure how a function changes as one variable changes, keeping the other variables constant.

Notation:

$$\text{Partial derivative of } f(x,y) \text{ with respect to } x: \quad \frac{\partial f}{\partial x}$$

$$\text{Partial derivative of } f(x,y) \text{ with respect to } y: \quad \frac{\partial f}{\partial y}$$

Lecture 9,10

25 How to Compute Partial Derivatives

- Treat all other variables as constants.
- Differentiate normally with respect to the chosen variable.

Example: If $f(x,y) = 3x^2y + 2xy^2$,

$$\frac{\partial f}{\partial x} = 6xy + 2y^2$$

$$\frac{\partial f}{\partial y} = 3x^2 + 4xy$$

26 Higher Order Partial Derivatives

We can take partial derivatives again to get higher-order derivatives.

Second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}$$

Mixed partial derivatives:

$$\frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

By Clairaut's theorem (if second derivatives are continuous),

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

27 Geometric Interpretation

- $\frac{\partial f}{\partial x}$ gives the slope of the surface in the x-direction.
- $\frac{\partial f}{\partial y}$ gives the slope in the y-direction.

Thus, partial derivatives describe how the surface tilts as you move in different directions.

28 Applications

- Finding tangent planes to surfaces.
- Solving problems in physics (e.g., heat flow, wave equations).
- Optimization problems with constraints (e.g., Lagrange multipliers).
- Gradient computations in machine learning.

29 Quick Example

Given $f(x, y) = x^2y + y^3$, find the first-order partial derivatives.

Solution:

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 + 3y^2$$

Summary

- Partial derivatives are like regular derivatives but for multi-variable functions.
- Keep other variables constant when differentiating.
- They are essential in higher mathematics, physics, engineering, and data science.

Lecture 11,12

30 Exact and Inexact Differentials

30.1 Exact Differentials

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is called **exact** if there exists a function $f(x, y)$ such that:

$$df = M(x, y)\,dx + N(x, y)\,dy$$

That is, M and N are partial derivatives of some scalar function f:

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

Condition for Exactness: If M and N have continuous first partial derivatives in a region R, then:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

is the necessary and sufficient condition for exactness.

30.2 Inexact Differentials

If there is no function $f(x, y)$ such that $df = M\,dx + N\,dy$, then the differential is called **inexact**.

In inexact cases:

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

31 Integrating Factor

An **integrating factor** is a function $\mu(x, y)$ such that multiplying the inexact differential makes it exact:

$$\mu(x, y)\,(M\,dx + N\,dy)$$

becomes exact.

Thus, after multiplication:

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Finding integrating factors generally requires intuition or specific techniques.

32 Simple Illustration

32.1 Example of Exact Differential

Given:

$$(2xy + 3)\,dx + (x^2 + 4)\,dy = 0$$

Check:

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the differential is **exact**.

Now, find $f(x, y)$:

$$\frac{\partial f}{\partial x} = 2xy + 3$$

Integrate w.r.t. x:

$$f(x, y) = x^2 y + 3x + g(y)$$

where $g(y)$ is a function of y only.

Now differentiate $f(x, y)$ w.r.t. y:

$$\frac{\partial f}{\partial y} = x^2 + g'(y)$$

Compare with $N = x^2 + 4$:

$$g'(y) = 4 \quad \Rightarrow \quad g(y) = 4y$$

Thus,

$$f(x, y) = x^2 y + 3x + 4y$$

The general solution is:

$$f(x, y) = C \quad \text{or} \quad x^2 y + 3x + 4y = C$$

32.2 Example of Inexact Differential and Integrating Factor

Given:

$$(y - x)\,dx + (y + x)\,dy = 0$$

Check:

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, this differential is actually **exact**.

Let us consider another:

Given:

$$(xy + y^2)\,dx + x^2\,dy = 0$$

Check:

$$\frac{\partial M}{\partial y} = x + 2y, \quad \frac{\partial N}{\partial x} = 2x$$

$$\text{Not equal} \quad \Rightarrow \quad \text{Inexact differential}$$

Try integrating factor $\mu = \frac{1}{x^2}$.

Multiplying through:

$$\left(\frac{xy + y^2}{x^2}\right) dx + \left(\frac{x^2}{x^2}\right) dy = \left(\frac{y}{x} + \frac{y^2}{x^2}\right) dx + dy$$

Now, check if it becomes exact:

$$M = \frac{y}{x} + \frac{y^2}{x^2}, \quad N = 1$$

$$\frac{\partial M}{\partial y} = \frac{1}{x} + \frac{2y}{x^2}, \quad \frac{\partial N}{\partial x} = 0$$

Still not exact. Another integrating factor would be needed depending on the situation.

(Thus, finding integrating factors sometimes involves guesswork or known techniques.)