4 Matrices

(a) Addition and Multiplication of matrices. Null Matrices. Diagonal, Scalar and Unit Matrices. Transpose of Matrix. Symmetric and Skew-Symmetric Matrices. Conjugate of a Matrix. Hermitian and Skew-Hermitian Matrices. Singular and Non-Singular Matrices. Orthogonal and Unitary Matrices. Trace of a Matrix. Inner Product.

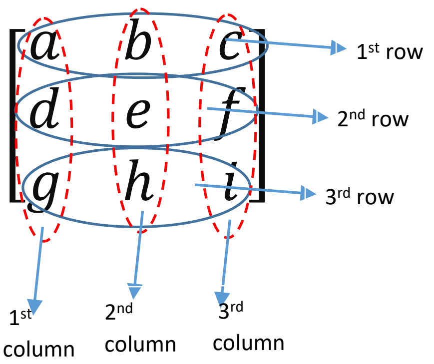
(b) Eigen-values and Eigen vectors. Cayley-Hamilton Theorem. Diagonalization of Matrices. Solution of Coupled Linear Ordinary Differential Equations. Function of a Matrix.

Part-(a)

Preliminaries

A matrix may be defined as a square or rectangular array of numbers (real or complex) or functions (real or complex) that obey certain laws. Thus the rectangular arrays $\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}, \begin{bmatrix} \sin\theta & \cos\theta \\ \pm\cos\theta & \mp\sin\theta \end{bmatrix}, \begin{bmatrix} 1+i2 & 3-i4 \\ 5+id & x+iy \\ s+i4t & 4+i7 \end{bmatrix}, \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_4(x) & f_5(x) & f_6(x) \end{bmatrix}$ are complex of matrix.

The individual members of the array are called the element of the matrix.



It is convenient to think of every element of a matrix are belonging to a certain row and a certain column.

If a matrix has m rows and n column, we shall say the matrix is order of $(m \times n)$ [read m by n]. A general matrix of order $(m \times n)$ can be conveniently written as,

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

This may be consider to the short hand notation $\boldsymbol{A} = [a_{ij}]_{m\times n}$ when $1 \le i \le m$ & $1 \le j \le n$. Where a_{ij} =the matrix element corresponding to i^{th} row and j^{th} column.

If the numbers of row and column is same, the matrix is referred to a square matrix, if otherwise, it is a rectangular matrix.

The determinants having same elements is denoted as $|A|$. In order that two matrices are equal to each other, it is necessary, though not sufficient, that they be same order. We say A is equal to B if and only if every element of A is equal to corresponding element of B i.e. $A = B$ if $a_{ij} = b_{ij}$ $1 \le i \le m$, $1 \le j \le n$

- **Row Matrix**: A matrix having single row only is called a row matrix. Thus $\boldsymbol{A} = [a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n}]$ is a row matrix.
- **Column Matrix**: A matrix having a column only is called a column matrix. $\boldsymbol{A} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} = \{a_{11} \quad a_{21} \quad a_{31} \quad \cdots \quad a_{m1}\}$ is a column matrix.
- **Diagonal Matrix**: A matrix whose principal diagonal elements only survive is called a diagonal matrix. Any matrix by suitable transformation can be thrown into a diagonal matrix. The process is known as diagonalisation. $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is a diagonal matrix.
- **Unit Matrix:** The unit matrix of order n contains diagonal elements as using. Thus $\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.
- **Null Matrix:** if a matrix contains all its elements are zero it is called a null matrix and is denoted as $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- **Singular Matrix**: If the determinant $|\boldsymbol{A}|$ of a matrix $\boldsymbol{A}$ is equal to zero, then matrix $\boldsymbol{A}$ is said to be a singular matrix. Thus $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a singular matrix.

ADDITION OF MATRICES: Here the both matrices must be of same order. Let $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}$. Then $a_{ij}+b_{ij}=c_{ij}$

Note that matrix addition is (i) Commutative [i.e., $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$] (ii) Associative [i.e., $\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$]

But these properties do not hold for Matrix subtraction.

MATRIX MULTIPLICATION: The Matrix product of $\boldsymbol{A}$ & $\boldsymbol{B}$ is defined when the number of columns of $\boldsymbol{A}$ be equal to the number of rows $\boldsymbol{B}$. For example, let $\boldsymbol{A}=[a_{ij}]_{m\times n}$ & $\boldsymbol{B}=[b_{ij}]_{n\times p}$. Then $\boldsymbol{AB}$ is defined but note that $\boldsymbol{BA}$ is not defined (undefined).

Let $=[\boldsymbol{a_{ij}}]_{m\times n}$, $\boldsymbol{B}=[b_{ij}]_{n\times p}$ & $\boldsymbol{C}=\boldsymbol{AB}$. Then $C_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$ where $1\le i\le m$, $1\le j\le p$

Note that $\boldsymbol{C}$ has m rows and p columns, $\boldsymbol{C}=[c_{ij}]_{m\times p}$.

Note that matrix multiplication may be or may not be commutative, but matrix multiplication is always associative. $(\boldsymbol{AB})\boldsymbol{C}=\boldsymbol{A}(\boldsymbol{BC})$.

Q0. **Proof (Commutative):** Let $\boldsymbol{A}=[a_{ij}]_{m\times n}$ & $\boldsymbol{B}=[b_{ij}]_{n\times p}$ then $\boldsymbol{AB}$ is defined but note that $\boldsymbol{BA}$ undefined. Hence in general matrix multiplication is not commutative.

Q1. **Show that matrix multiplication is in general associative.**

$\Rightarrow$ $\boldsymbol{A}=[a_{ij}]_{m\times n}$, $\boldsymbol{B}=[b_{ij}]_{n\times p}$ & $\boldsymbol{C}=[c_{ij}]_{p\times q}$

$[(AB)C]_{ij}=\sum_{k=1}^{p}(AB)_{ik}C_{kj}=\sum_{k=1}^{p}\sum_{l=1}^{n}(A)_{il}(B)_{lk}(C)_{kj}=\sum_{k=1}^{p}\sum_{l=1}^{n}a_{il}\,b_{lk}c_{kj}$

$[A(BC)]_{ij}=\sum_{l=1}^{n}A_{il}(BC)_{lj}=\sum_{l=1}^{n}(A)_{il}\sum_{k=1}^{p}(B)_{lk}(C)_{kj}=\sum_{l=1}^{n}\sum_{k=1}^{p}a_{il}\,b_{lk}c_{kj}$

Q2. **Show that matrix multiplication is distributive with respect to matrix addition i.e. $A(B+C)=AB+AC$**

$\Rightarrow$ $\boldsymbol{A}=[a_{ij}]_{m\times n}$, $\boldsymbol{B}=[b_{ij}]_{n\times p}$ & $\boldsymbol{C}=[c_{ij}]_{p\times q}$

$[A(B+C)]_{ij}=\sum_{k=1}^{n}(A)_{ik}(B+C)_{kj}=\sum_{k=1}^{n}(A)_{ik}\{(B)_{kj}+(C)_{kj}\}=\sum_{k=1}^{n}a_{ik}\{b_{kj}+c_{kj}\}$

$[(AB)+(AC)]_{ij}=(AB)_{ij}+(AC)_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}+\sum_{k=1}^{n}a_{ik}c_{kj}=\sum_{k=1}^{n}a_{ik}\{b_{kj}+c_{kj}\}$

Q2.1, AB=BA Prove that A & B are square matrix of same dimension.

$\Rightarrow$ Let us consider $\boldsymbol{A}=[a_{ij}]_{m\times n}$ and $\boldsymbol{B}=[b_{ij}]_{p\times q}$

For conformable for product $\boldsymbol{AB}$ we must have $n=p$. Again for conformable for product $\boldsymbol{BA}$ we must have $q=m$.

Now $\boldsymbol{AB}=[c_{ij}]_{m\times q}$ and $\boldsymbol{BA}=[d_{ij}]_{p\times n}$ as $\boldsymbol{AB}=\boldsymbol{BA}$ then we must have $m=p$ & $q=n$

Then $m=n=p=q$. Hence $\boldsymbol{A}$ & $\boldsymbol{B}$ are square matrix of same dimension

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- **Diagonal Matrix**: $\boldsymbol{A}=[a_{ij}]_{n\times n}$, diagonal matrix is a square matrix for which $\begin{matrix} a_{ij}\neq 0 & for\ i=j \\ a_{ij}=0 & for\ i\neq j\end{matrix}$. $\boldsymbol{A}=\begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}$
- **Scalar Matrix:** A diagonal matrix for which all the non-vanishing elements are equal to each other. i.e. $\begin{matrix} a_{ij}=c & for\ i=j \\ a_{ij}=0 & for\ i\neq j\end{matrix}$.
  $\boldsymbol{A}=\begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}$
- **Unit/Identity Matrix:** When $\begin{matrix} a_{ij}=1 & for\ i=j \\ a_{ij}=0 & for\ i\neq j\end{matrix}$ i.e. $a_{ij}=\delta_{ij}$ where $\begin{matrix} a_{ij}=1 & for\ i=j \\ a_{ij}=0 & for\ i\neq j\end{matrix}$ [δ=Kronecker delta]

Q3. If a matrix $\boldsymbol{B}$ commutes with a diagonal matrix no two diagonal elements of which are equal to each other, show that $\boldsymbol{B}$ must be a diagonal matrix.

$\Rightarrow$ Let a diagonal matrix $\boldsymbol{A}$, then $[A]_{ij}=a_i\delta_{ij}$. Where a_i are scalar such that $a_i\neq a_j$ and $1\le i,j\le n$

Now we have $\boldsymbol{AB}=\boldsymbol{BA}$ $\Rightarrow \sum_k (A)_{ik}(B)_{kj}=\sum_k (B)_{ik}(A)_{kj}$

$or, \sum_k a_i\,\delta_{ik}b_{kj}=\sum_k b_{ik}a_k\delta_{kj}$

or, $a_ib_{ij}=b_{ij}a_j$ or, $b_{ij}(a_i-a_j)=0$ for $i\neq j$, $a_i\neq a_j$ then $b_{ij}=0$ for $i\neq j$. **Then $\boldsymbol{B}$ must be a diagonal matrix.**

Transpose of a Matrix: $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ then $\boldsymbol{A^T} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$. If $\boldsymbol{A} = [a_{ij}]_{m\times n}$ then $\boldsymbol{A^T} = [a_{jj}]_{n\times m}$.

Conjugate of a Matrix: If $\boldsymbol{A} = [a_{ij}]_{m\times n}$ then $\boldsymbol{A}^* = [a_{ij}{}^*]_{m\times n}$.

Hermitian Conjugate Matrix: $\boldsymbol{A} = [a_{ij}]_{m\times n}$ then $[\boldsymbol{A}^*]^{\boldsymbol{T}} \equiv [\boldsymbol{A^T}]^* \equiv \boldsymbol{A}^\dagger$ is called Hermitian conjugate. $\boldsymbol{A}^\dagger = [a_{ji}{}^*]_{n\times m}$.

Symmetric & Skew-symmetric Matrices: $\boldsymbol{A^T} = \boldsymbol{A}$ then symmetric. $\boldsymbol{A^T} = -\boldsymbol{A}$ then skew-symmetric.

Hence $a_{ji} = a_{ij}$ symmetric, $a_{ji} = -a_{ij}$ skew-symmetric.

Real and Purely Imaginary Matrices: $\boldsymbol{A}^* = \boldsymbol{A}$ then real. $\boldsymbol{A}^* = -\boldsymbol{A}$ Then purely imaginary. Hence $a_{ij}{}^* = a_{ij}$ real and $a_{ij}{}^* = -a_{ij}$ purely imaginary.

Hermitian & Skew-Hermitian Matrices: $\boldsymbol{A}^\dagger = \boldsymbol{A}$ then Hermitian. $\boldsymbol{A}^\dagger = -\boldsymbol{A}$ then skew-Hermitian.

Hence $a_{ji}{}^* = a_{ij}$ Hermitian, $a_{ji}{}^* = -a_{ij}$ skew-Hermitian.

Orthogonal Matrix: When $\boldsymbol{AA^T} = \boldsymbol{A^TA} = \boldsymbol{I} \Rightarrow \boldsymbol{A^T} = \boldsymbol{A^{-1}}$ (I is identity matrix). Equating the ij^{th} element $\sum_{k=1}^{n} a_{ik}a_{jk} = \delta_{ij}\ for\ 1 \le i,j \le n$ $or, \sum_{k=1}^{n} a_{ki}a_{kj} = \delta_{ij}\ for\ 1 \le i,j \le n$

Unitary Matrix: When $\boldsymbol{AA}^\dagger = \boldsymbol{A}^\dagger\boldsymbol{A} = \boldsymbol{I} \Rightarrow \boldsymbol{A}^\dagger = \boldsymbol{A^{-1}}$. Equating the ij^{th} element $\sum_{k=1}^{n} a_{ik}a^*{}_{jk} = \delta_{ij}\ for\ 1 \le i,j \le n$ $or, \sum_{k=1}^{n} a_{ki}a^*{}_{kj} = \delta_{ij}\ for\ 1 \le i,j \le n$

Inverse of a matrix: if A≡ $[a_{ij}]_{n\times n}$ be a nonsingular square matrix then it's inverse matrix **R**≡ $[R_{ij}]_{n\times n}$ is defined as $\boldsymbol{AR} = \boldsymbol{I}$ (I is identity matrix)

The defining equation for R is **R**= $\frac{[a^{ji}]_{n\times n}}{|\boldsymbol{A}|} = \frac{adj(\boldsymbol{A})}{|\boldsymbol{A}|}$ where $[a^{ji}]_{n\times n}$ is transpose of co-factor of matrix **A**=adjoin of matrix **A**, $|\boldsymbol{A}|$ →Determinant of **A.**

NOTE THAT:- let A≡ $[a_{ij}]_{n\times n}$ is a square matrix. Then $\boldsymbol{M_{ij}}$=$[m_{ij}]_{n-1\times n-1}$ the **sub-matrix** of obtained by deleting it's i[th] row and j[th] column. And $|\boldsymbol{M_{ij}}|$ is called minor of element a_{ij} of **A**

Co-factor of a matrix A≡ $[a_{ij}]_{n\times n}$ is defined as $\mathbf{A^C} = [a^{ij}]_{n\times n}$ where $a^{ij} = (-1)^{i+j}|\boldsymbol{M_{ij}}|$

i.e,if **A**≡ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\mathrm{A}^C = \begin{bmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{bmatrix} = \begin{bmatrix} (-1)^{1+1}|\boldsymbol{M_{11}}| & (-1)^{1+2}|\boldsymbol{M_{12}}| & (-1)^{1+3}|\boldsymbol{M_{13}}| \\ (-1)^{2+1}|\boldsymbol{M_{21}}| & (-1)^{2+2}|\boldsymbol{M_{22}}| & (-1)^{2+3}|\boldsymbol{M_{23}}| \\ (-1)^{3+1}|\boldsymbol{M_{31}}| & (-1)^{3+2}|\boldsymbol{M_{32}}| & (-1)^{3+3}|\boldsymbol{M_{33}}| \end{bmatrix}$

Adjoin matrix: The transpose of co-factor of matrix **A** is called the adjoin of matrix **A.**

Adjoin of a matrix $\boldsymbol{A} \equiv [a_{ij}]_{m\times n}$ is defined as $adj(\boldsymbol{A}) = [a'_{ij}]_{n\times n}$ where $a'_{ij} = (-1)^{j+i}|\boldsymbol{M_{ji}}|$

Singular Matrix: If the determinant $|\boldsymbol{A}|$ of a matrix $\boldsymbol{A}$ is equal to zero, then matrix $\boldsymbol{A}$ is said to be a singular matrix. Thus $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a singular matrix.

Hence inverse of matrix exists only for nonsingular matrix.

NOTE THAT: A square matrix $\boldsymbol{A}$ for which $\boldsymbol{A}^{K+1} = \boldsymbol{A}$ is said to be **Periodic matrix** and the least positive value of K satisfying this relation is called period of the periodicity.

When the period of periodicity is unity that means when $\boldsymbol{A^{1+1}} = \boldsymbol{A^2} = \boldsymbol{A}$ then matrix is called **Idempotent matrix.**

When there exists a positive integer p is such that $\boldsymbol{A}^p = 0$ (null matrix), then A is called **Nill-potent matrix**. And p is called index of nill-potency.

When $\boldsymbol{A^2} = \boldsymbol{I}$ then A is called **Involuntary matrix.**

Q4. Show that $(A^T)^T = A$. $\Rightarrow$ Let $A = [a_{ij}]_{m\times n}$ then $A^T = [a_{ji}]_{n\times m}$. So, $(A^T)^T = [a_{ij}]_{m\times n} = A$ (proved).

Q5. Show that $(AB)^T = B^T A^T$.

$\Rightarrow$ Let $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{n\times p}$ then $A^T = [a_{ji}]_{n\times m}$ & $B^T = [b_{ji}]_{p\times n}$. Now the order of AB is $m\times p$ and $(AB)^T$ is $p\times m$. So the order of $B^T A^T$ is $p\times m$. Then order of $(AB)^T$ & $B^T A^T$ are conformable.

$[(AB)^T]_{ij} = (AB)_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$ ……(i)

$(B^T A^T)_{ij} = \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} = \sum_{k=1}^{n} (B)_{ik} (A)_{kj} = \sum_{k=1}^{n} b_{ki} a_{jk} = \sum_{k=1}^{n} a_{jk} b_{ki}$ ……(ii) $\therefore (AB)^T = B^T A^T$

Q6. Home work: (i) $(\lambda A)^T = \lambda A^T$ **when λ is scalar.** (ii) $(A+B)^T = A^T + B^T$.

Q7. Show that every square matrix can be uniquely written as the sum of a symmetric and anti-symmetric matrix. $A = \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T]$.

$\Rightarrow$ Let $A = \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T] = P + Q$

$P^T = \left\{\frac{1}{2}(A + A^T)\right\}^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T) = P$ and $Q^T = \left\{\frac{1}{2}(A - A^T)\right\}^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -Q$

Then P matrix is symmetric but Q matrix is skew-symmetric. Let $A = R + S$ where R is symmetric & S is skew-symmetric.

Then $A^T = (R+S)^T = R^T + S^T = R - S$. So, $A = R + S$ & $A^T = R - S$ $\Rightarrow$ $A + A^T = 2R$ and $A - A^T = 2S$

$\therefore R = \frac{1}{2}(A^T + A)$ similarly $S = \frac{1}{2}(A - A^T)$.

Hence representation of $A = \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T]$ is unique.

Q8. Show that all diagonal elements of skew-symmetric matrix are zero.

$\Rightarrow$ For skew-symmetric matrix $a_{ij} = -a_{ji}$ for all value of i & j. And for diagonal elements $j = i$ i.e. $a_{ii} = -a_{ii}$. Hence $2a_{ii} = 0 \Rightarrow a_{ii} = 0$

Hence all diagonal elements are zero. i.e. $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is skew-symmetric.

Q9. Show that the product of two symmetric matrix is symmetric if and only if they commute.

$\Rightarrow$ We have $A^T = A$ and $B^T = B$. Now $(AB)^T = B^T A^T = BA = AB$ if A & B commute.

Q10. Show that AA^T is a symmetric matrix using index notation.

$\Rightarrow \{(AA^T)^T\}_{ij} = (AA^T)_{ji} = \sum (A)_{jk} (A^T)_{ki} = \sum a_{jk} a_{ik}$ ... (i) and $(AA^T)_{ij} = \sum (A)_{ik} (A^T)_{kj} = \sum a_{ik} a_{jk} = \sum a_{jk} a_{ik}$ ... (ii) Hence $(AA^T)^T = AA^T$.

Q11. If A is an anti-symmetric matrix of order n and X is a column vector/matrix of dimension, show that $X^T A X = 0$.

$\Rightarrow$ We have $A^T = -A$. Now $(X^T A X)^T = X A^T X^T = -(X^T A X)$ ... (i)

But $X^T A X$ has just one row and one column, hence it is transpose must be identical to itself i.e. $(X^T A X)^T = X^T A X$ ... (ii)

From (i) & (ii) we have $X^T A X = -(X^T A X)$ i.e. $X^T A X = 0$ (proved).

Q12. Show that $(AB)^* = A^* B^*$.

$\Rightarrow A^* B^* = \sum_k a_{ik}^* b_{kj}^*$ again $AB = \sum_k a_{ik} b_{kj}$. Then $(AB)^* = \sum_k a_{ik}^* b_{kj}^* = A^* B^*$ (proved).

Q13. Home work: (i)$(A^*)^* = A$, (ii) $(A+B)^* = A^* + B^*$, (iii) $(\lambda A)^* = \lambda^* A^*$.

Q14. Show that, $(AB)^\dagger = B^\dagger A^\dagger$.

$\Rightarrow$Let, Let $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{n\times p}$ then $A^\dagger = [a^*{}_{ji}]_{n\times m}$ & $B^\dagger = [b^*{}_{ji}]_{p\times n}$. Now the order of $B^\dagger A^\dagger$ is $p\times m$ where $(AB)^\dagger$ is a matrix of order $p\times m$. Then order of $(AB)^\dagger$ & $B^\dagger A^\dagger$ are conformable.

$[(AB)^\dagger]_{ij} = [(AB)^*]_{ji} = \sum_{k=1}^{n} a^*{}_{jk} b^*{}_{ki}$ ……(i). Again,

$(B^\dagger A^\dagger)_{ij} = \sum_{k=1}^{n} (B^\dagger)_{ik} (A^\dagger)_{kj} = \sum_{k=1}^{n} (B^*)_{ik} (A^*)_{kj} = \sum_{k=1}^{n} a^*{}_{jk} b^*{}_{ki}$ ……(ii) $\therefore (AB)^\dagger = B^\dagger A^\dagger$

Q15. Home work: (i)$(A^\dagger)^\dagger = A$, (ii) $(A+B)^\dagger = A^\dagger + B^\dagger$, (iii) $(\lambda A)^\dagger = \lambda^* A^\dagger$.

Q16. Show that every square matrix can be uniquely written as the sum of a Hermitian and anti-Hermitian matrix.

⇒ Let A be a square matrix, the evidently the matrix can be represented as, $\boldsymbol{A} = \frac{1}{2}[\boldsymbol{A}+\boldsymbol{A}^\dagger] + \frac{1}{2}[\boldsymbol{A}-\boldsymbol{A}^\dagger] = \boldsymbol{P}+\boldsymbol{Q}$ (say)

$\boldsymbol{P}^\dagger = \left\{\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^\dagger)\right\}^\dagger = \frac{1}{2}(\boldsymbol{A}^\dagger+\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^\dagger) = \boldsymbol{P}$ and $\boldsymbol{Q}^\dagger = \left\{\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^\dagger)\right\}^\dagger = \frac{1}{2}(\boldsymbol{A}^\dagger-\boldsymbol{A}) = -\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^\dagger) = -\boldsymbol{Q}$

Then $\boldsymbol{P}$ is Hermitian matrix but $\boldsymbol{Q}$ is anti-Hermitian matrix. Let $\boldsymbol{A}=\boldsymbol{R}+\boldsymbol{S}$ where $\boldsymbol{R}$ is Hermitian matrix & $\boldsymbol{S}$ is skew-Hermitian matrix.

Then $\boldsymbol{A}^\dagger = (\boldsymbol{R}+\boldsymbol{S})^\dagger = \boldsymbol{R}^\dagger+\boldsymbol{S}^\dagger = \boldsymbol{R}-\boldsymbol{S}$. So, $\boldsymbol{A}=\boldsymbol{R}+\boldsymbol{S}$ & $\boldsymbol{A}^\dagger=\boldsymbol{R}-\boldsymbol{S}$ ⇒ $\boldsymbol{A}+\boldsymbol{A}^\dagger = 2\boldsymbol{R}$ and $\boldsymbol{A}-\boldsymbol{A}^\dagger = 2\boldsymbol{S}$

$\therefore \boldsymbol{R} = \frac{1}{2}(\boldsymbol{A}^\dagger+\boldsymbol{A})$ similarly $\boldsymbol{S} = \frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^\dagger)$.

Hence representation of $\boldsymbol{A} = \frac{1}{2}[\boldsymbol{A}+\boldsymbol{A}^\dagger] + \frac{1}{2}[\boldsymbol{A}-\boldsymbol{A}^\dagger]$ is unique.

Q17. Show that every diagonal element of a skew-Hermitian matrix either zero or a pure imaginary number.

⇒ Any square matrix is skew-Hermitian if $a_{ij} = -a_{ji}^*$. For diagonal element $i=j$ then $a_{ii} = -a_{ii}^*$. i.e. $a_{ii}+a_{ii}^* = 0 \ldots(i)$

Then if $a_{ii} = x+iy$ then $a_{ii}^* = x-iy$ then $a_{ii}+a_{ii}^* = 2x \ldots(ii)$. Hence from (i) & (ii) x is always zero then $a_{ii} = iy$ always.

Hence every diagonal elements of a skew-Hermitian matrix either zero or pure imaginary number.

Q18. Show that every diagonal elements of a Hermitian matrix are REAL.

⇒ If any square matrix is Hermitian then, $a_{ij} = -a_{ji}^*$. For diagonal element $i=j$ then $a_{ii} = -a_{ii}^*$ for all value of i. Hence diagonal elements of Hermitian matrix are REAL.

Q19. If $\boldsymbol{A}$ is a Hermitian matrix, then show that $\boldsymbol{B}^\dagger\boldsymbol{A}\boldsymbol{B}$ is Hermitian for any matrix $\boldsymbol{B}$.

⇒ A is Hermitian, then $A^\dagger = A$. Now $(\boldsymbol{B}^\dagger\boldsymbol{A}\boldsymbol{B})^\dagger = \boldsymbol{B}^\dagger\boldsymbol{B}\boldsymbol{A}^\dagger(\boldsymbol{B}^\dagger)^\dagger = \boldsymbol{B}^\dagger\boldsymbol{A}\boldsymbol{B}$ for any matrix $\boldsymbol{B}$.

Q20. If $\boldsymbol{H}$ is a Hermitian matrix, then show that $\boldsymbol{iH}$ is skew-Hermitian matrix.

⇒ $\boldsymbol{H}^\dagger = \boldsymbol{H}$ as $\boldsymbol{H}$ is Hermitian. Now, $(\boldsymbol{iH})^\dagger = \boldsymbol{i}^*\boldsymbol{H}^\dagger = -\boldsymbol{iH}$ (proved).

Q21. Home work: $\boldsymbol{H_1}$ is Hermitian and $\boldsymbol{H_2}$ is anti-Hermitian, show that $\boldsymbol{H_1}+\boldsymbol{iH_2}$ and $\boldsymbol{H_1}-\boldsymbol{iH_2}$ both are Hermitian.

Q22. Determinant of an orthogonal matrix can only have values $+1$ *or* -1.

⇒ Let $det\boldsymbol{A} = d$ then $det\boldsymbol{A}^{\boldsymbol{T}} = d$ where $\boldsymbol{A}$ is an orthogonal matrix. We have then $d^2=1$ as $\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{I}$ or, $d=\pm 1$

Note that, the elements of an orthogonal matrix may be real or complex. However in practice, we are always concerned with real orthogonal matrices.

Q23. if a matrix A is Hermitian and $A^2 = I$. Show that it is also unitary.

Ans: $A^2 = I$ then $AA = I$ or, $AA^\dagger = I$ (proved) [as A is Hermitian then $A = A^\dagger$]

Q24. Determinant of an unitary matrix can be complex number of unit magnitude.

⇒ Let $det\boldsymbol{A} = d$ then $det\boldsymbol{A}^\dagger = d^*$ where $\boldsymbol{A}$ is an unitary matrix. We have then $dd^* = 1\ or, |d|^2 = 1$. i.e. unitary matrix is non-singular matrix.

Note that, the elements of an unitary matrix may be complex. In fact it is evident that real unitary matrix is orthogonal.

Q25. Obtain the must general orthogonal matrix of order 2.

⇒ Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. So, $\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$. Comparing with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we have $\begin{matrix} a^2+b^2=1 \\ ac+bd=0 \\ c^2+d^2=1 \end{matrix}$….(i)

Hence equation (i) shows, (a) sum of squares of the elements of any row is equal to unity. (b) The sum of product of corresponding elements of two distinct rows of an orthogonal matrix is zero.

If we choose $\begin{matrix} a=\cos\theta & b=\sin\theta \\ c=\cos\varphi & d=\sin\varphi \end{matrix}$ then $ac+bd=0$ ⇒ $\cos(\theta-\varphi)=0\ or, \theta-\varphi = \pm\frac{\pi}{2}$ $\therefore \varphi = \theta\pm\frac{\pi}{2}$

Then $c = \mp\sin\theta$ & $d = \mp\cos\theta$. Then $\boldsymbol{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ \mp\sin\theta & \mp\cos\theta \end{bmatrix}$. Which is the most general form of orthogonal matrix of order 2.

Q26. Obtain the must general unitary matrix of order 2.

$\Rightarrow$ Let $\boldsymbol{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\boldsymbol{A}^{\dagger}=\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$. So, $\boldsymbol{A}\boldsymbol{A}^{\dagger}=\begin{bmatrix} aa^*+bb^* & ac^*+bd^* \\ a^*c+bd^* & cc^*+dd^* \end{bmatrix}$. Comparing with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we have $\begin{matrix} |a|^2+|b|^2=1 \\ ac^*+bd^*=0 \\ |c|^2+|d|^2=1 \end{matrix}$….(i)

We can choose a and b to be of the form. $a=e^{i\alpha}\cos\theta$ & $b=e^{i\gamma}\sin\theta$. Hence $\frac{c^*}{d^*}=-\frac{e^{i\gamma}\sin\theta}{e^{i\alpha}\cos\theta}$……(ii)

In order to satisfy the second of equation (i) and (ii) we can choose $c=-e^{i(\beta-\gamma)}\sin\theta$ & $d=e^{i(\beta-\alpha)}\cos\theta$ [β real]

Then $\boldsymbol{A}=\begin{bmatrix} e^{i\alpha}\cos\theta & e^{i\gamma}\sin\theta \\ -e^{i(\beta-\gamma)}\sin\theta & e^{i(\beta-\alpha)}\cos\theta \end{bmatrix}$ Which involves four real independent parameter $\alpha,\beta,\gamma,\theta$. It is the most general form of unitary matrix order 2.

Q27. Show that for any square matrix A, A (adjA)=(adj A) A= |A|*I*

$$\boldsymbol{A}\,(adj\boldsymbol{A})=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} (-1)^{1+1}|\boldsymbol{M_{11}}| & (-1)^{2+1}|\boldsymbol{M_{21}}| & (-1)^{3+1}|\boldsymbol{M_{31}}| \\ (-1)^{1+2}|\boldsymbol{M_{12}}| & (-1)^{2+2}|\boldsymbol{M_{22}}| & (-1)^{3+2}|\boldsymbol{M_{3+2}}| \\ (-1)^{1+3}|\boldsymbol{M_{13}}| & (-1)^{2+3}|\boldsymbol{M_{23}}| & (-1)^{3+3}|\boldsymbol{M_{33}}| \end{bmatrix}$$

$$=\begin{bmatrix} \sum_{j=1}^{n}(-1)^{1+j}a_{1j}|\boldsymbol{M_{1j}}| & \sum_{j=1}^{n}(-1)^{1+j}a_{1j}|\boldsymbol{M_{2j}}| & \sum_{j=1}^{n}(-1)^{1+j}a_{1j}|\boldsymbol{M_{3j}}| \\ \sum_{j=1}^{n}(-1)^{2+j}a_{2j}|\boldsymbol{M_{1j}}| & \sum_{j=1}^{n}(-1)^{2+j}a_{2j}|\boldsymbol{M_{2j}}| & \sum_{j=1}^{n}(-1)^{2+j}a_{1j}|\boldsymbol{M_{3j}}| \\ \sum_{j=1}^{n}(-1)^{3+j}a_{3j}|\boldsymbol{M_{1j}}| & \sum_{j=1}^{n}(-1)^{3+j}a_{3j}|\boldsymbol{M_{2j}}| & \sum_{j=1}^{n}(-1)^{3+j}a_{1j}|\boldsymbol{M_{3j}}| \end{bmatrix}=\begin{bmatrix} |\boldsymbol{A}| & 0 & 0 \\ 0 & |\boldsymbol{A}| & 0 \\ 0 & 0 & |\boldsymbol{A}| \end{bmatrix}=|\boldsymbol{A}|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=|\boldsymbol{A}|\boldsymbol{I}$$

Q28. If A is a non-singular matrix of order n, then $|adjA|=|A|^{n-1}$.

$\Rightarrow \boldsymbol{A}\,(adj\boldsymbol{A})=|\boldsymbol{A}|\boldsymbol{I}\;\; or,\; |\boldsymbol{A}\,(adj\boldsymbol{A})|=\big||\boldsymbol{A}|\boldsymbol{I}\big|\; or, |\boldsymbol{A}||adj\boldsymbol{A}|=|\boldsymbol{A}|^n\;\; or,\; |adj\boldsymbol{A}|=|\boldsymbol{A}|^{n-1}$

Q29. If A & B are two square matrices of order n then $(adj\boldsymbol{AB})=(adj\boldsymbol{B})(adj\boldsymbol{A})$.

$\Rightarrow$ Therefore $(\boldsymbol{AB})(adj\boldsymbol{AB})=(adj\boldsymbol{AB})(\boldsymbol{AB})=|\boldsymbol{AB}|\boldsymbol{I}$

Consider $(\boldsymbol{AB})(adj\boldsymbol{B}\;adj\boldsymbol{A})$, $(\boldsymbol{AB})(adj\boldsymbol{B}\;adj\boldsymbol{A})=A(\boldsymbol{B}\;adj\boldsymbol{B})adj\boldsymbol{A}=\boldsymbol{A}(|\boldsymbol{B}|\boldsymbol{I})adj\boldsymbol{A}=|\boldsymbol{B}|(\boldsymbol{A}adj\boldsymbol{A})=|\boldsymbol{B}||\boldsymbol{A}|\boldsymbol{I}=|\boldsymbol{A}||\boldsymbol{B}|\boldsymbol{I}=|\boldsymbol{AB}|\boldsymbol{I}$….(i)

Also consider, $(adj\boldsymbol{B}\;adj\boldsymbol{A})(\boldsymbol{AB})$, $(adj\boldsymbol{B}\;adj\boldsymbol{A})(\boldsymbol{AB})=adj\boldsymbol{B}(adj\boldsymbol{A}\;\boldsymbol{A})\boldsymbol{B}=adj\boldsymbol{B}\;|\boldsymbol{A}|\boldsymbol{I}\;\boldsymbol{B}=|\boldsymbol{A}|adj\boldsymbol{B}\;\boldsymbol{B}=|\boldsymbol{A}||\boldsymbol{B}|\boldsymbol{I}=|\boldsymbol{AB}|\boldsymbol{I}$….(ii)

Now from (i) & (ii) we conclude that, $(adj\boldsymbol{AB})=(adj\boldsymbol{B})(\,adj\boldsymbol{A})$

Note that, some results of adjoint: (i) for any square matrix $\boldsymbol{A}$ $(adj\boldsymbol{A})^T=adj\boldsymbol{A}^{\boldsymbol{T}}$**.** (ii) The adjoint of an identity matrix is the identity matrix. (iii) The adjoint of a symmetric matrix is a symmetric matrix.

Trace of a matrix: The sum of diagonal element of a matrix **A** is called trace of a matrix **A** i.e, $Tr(\boldsymbol{A})=\sum_{i=1}^{n}a_{ii}$ thus if **A**=$\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}$ **Tr(A) =10**

Q30. SHOW THAT $\boldsymbol{Tr}(\boldsymbol{AB})=\boldsymbol{Tr}(\boldsymbol{BA})$.

$\Rightarrow Tr(\boldsymbol{AB})=\sum_{i=1}^{n}(\boldsymbol{AB})_{ii}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}b_{ji}$ again $Tr(\boldsymbol{BA})=\sum_{j=1}^{n}(\boldsymbol{AB})_{jj}=\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ji}a_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}b_{ji}=Tr(\boldsymbol{AB})$ **(proved)**

Q31. Show that $\boldsymbol{Tr}(\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\cdots)=\boldsymbol{TrA}+\boldsymbol{TrB}+\boldsymbol{TrC}+\cdots$.

$\Rightarrow Tr(\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\cdots)=\sum_i(\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\cdots)_{\boldsymbol{ij}}=\sum(\boldsymbol{A})_{\boldsymbol{ii}}+(\boldsymbol{B})_{\boldsymbol{ii}}+(\boldsymbol{C})_{\boldsymbol{ii}}+\cdots=\sum(\boldsymbol{A})_{\boldsymbol{ii}}+\sum(\boldsymbol{B})_{\boldsymbol{ii}}+\sum(\boldsymbol{C})_{\boldsymbol{ii}}+\cdots=Tr\boldsymbol{A}+Tr\boldsymbol{B}+Tr\boldsymbol{C}+\cdots$ (proved)

Q32. Show that $\boldsymbol{Tr}[(\boldsymbol{AB})^{\boldsymbol{k}}]=\boldsymbol{Tr}[(\boldsymbol{BA})^{\boldsymbol{k}}]$.

$\Rightarrow Tr\left[(\boldsymbol{AB})^{\boldsymbol{k}}\right]=Tr\left[\boldsymbol{A}^{\boldsymbol{k}}\boldsymbol{B}^{\boldsymbol{k}}\right]=Tr(\boldsymbol{CD})$ Say $\boldsymbol{A}^{\boldsymbol{k}}=\boldsymbol{C}$ & $\boldsymbol{B}^{\boldsymbol{k}}=\boldsymbol{D}$ $\therefore Tr(\boldsymbol{DC})=Tr\left[\boldsymbol{B}^{\boldsymbol{k}}\boldsymbol{A}^{\boldsymbol{k}}\right]=Tr\left[(\boldsymbol{BA})^{\boldsymbol{k}}\right]$ (proved)

Inner Product: In mathematics, the **Frobenius inner product** is a binary operation that takes two matrices and returns a number. It is often denoted as $\langle\boldsymbol{A},\boldsymbol{B}\rangle_F$. The operation is a component-wise inner product of two matrices as though they are vectors. The two matrices must have the same dimension—same number of rows and columns—but are not restricted to be square matrices.

Given two complex number-valued $m\times n$ matrices **A** and **B**, written explicitly as

$$\boldsymbol{A} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad \& \ \boldsymbol{B} \equiv \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix}_{m\times n}$$

the Frobenius inner product is defined by the following summation Σ of matrix elements,

$\langle \boldsymbol{A},\boldsymbol{B}\rangle_F = \sum_{i.j}^{n} \overline{(A)}_{ij}\,(B)_{ij} = Tr(\overline{\boldsymbol{A}}^{\boldsymbol{T}}\boldsymbol{B})$

Where the over line denotes the complex conjugate. Explicitly this sum is $\langle \boldsymbol{A},\boldsymbol{B}\rangle_F = \begin{matrix} a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + \cdots .. a_{1m}b_{1n} \\ a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} + \cdots .. a_{2m}b_{2n} \\ a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33} + \cdots .. a_{3m}b_{3n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}b_{m1} + a_{m2}b_{n2} + a_{m3}b_{m3} + \cdots .. a_{nm}b_{mn} \end{matrix}$

Properties

It is a sesquilinear form, for four complex-valued matrices **A**, **B**, **C**, **D**, and two complex numbers a and b:(I) $\langle a\boldsymbol{A}, b\boldsymbol{B}\rangle_F = \bar{a}\, b\,\langle \boldsymbol{A},\boldsymbol{B}\rangle_F$

(II) $\langle \boldsymbol{A}+\boldsymbol{C}, \boldsymbol{B}+\boldsymbol{D}\rangle_F = \langle \boldsymbol{A},\boldsymbol{B}\rangle_F + \langle \boldsymbol{A},\boldsymbol{D}\rangle_F + \langle \boldsymbol{B},\boldsymbol{C}\rangle_F + \langle \boldsymbol{B},\boldsymbol{D}\rangle_F$

(III) Also, exchanging the matrices amounts to complex conjugation: $\langle \boldsymbol{A},\boldsymbol{B}\rangle_F = \overline{\langle \boldsymbol{A},\boldsymbol{B}\rangle}_F$

(IV) For the same matrix, $\langle \boldsymbol{A},\boldsymbol{A}\rangle_F \geq 0$

FOR REAL VALUED MATRIX: $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 6 \\ 1 & -1 & 2 \end{bmatrix}$ & $\boldsymbol{B} = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 1 & -5 \end{bmatrix}$ THEN $\langle \boldsymbol{A},\boldsymbol{B}\rangle_F = 16 + 0 + 12 + 4 - 1 - 10 = 21.$

FOR COMPLEX VALUED MATRIX **A**=$\begin{bmatrix} 1+i & -2i \\ 3 & -5 \end{bmatrix}$ & **B**= $\begin{bmatrix} -2 & 3i \\ 4-3i & 6 \end{bmatrix}$. $\overline{\boldsymbol{A}}$=$\begin{bmatrix} 1-i & 2i \\ 3 & -5 \end{bmatrix}$ & **B**= $\begin{bmatrix} -2 & 3i \\ 4-3i & 6 \end{bmatrix}$

$$\langle \boldsymbol{A},\boldsymbol{B}\rangle_F = -2(1-i) - 6 + 3(4-3i) - 30 = -2 - 6 + 12 - 30 + 2i - 9i = -26 - 7i$$

Note that if the matrices are vectorised then $\langle \boldsymbol{A},\boldsymbol{B}\rangle_F = vec(A)^T vec(B)$ Where $vec(A) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nm} \end{bmatrix}$ $vec(B) = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{nm} \end{bmatrix}$

Part-(b)

<u>Eigen Values and Eigen Vectors</u>

<u>Characteristic Matrix:</u> For a given square matrix $\boldsymbol{A}$, the matrix $\boldsymbol{A} - \boldsymbol{\lambda I}$ is called characteristic of $\boldsymbol{A}$, where λ is a scalar parameter and $\boldsymbol{I}$ is the unit matrix as same order.

Example: Let $\boldsymbol{A} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ then $\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore \boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$

Which is the required characteristic matrix.

<u>Characteristic polynomial:</u> The determinant $|\boldsymbol{A} - \boldsymbol{\lambda I}|$on expansion will give rise to polynomial and is known as characteristic polynomial of matrix $\boldsymbol{A}$.

Example: $\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = (6-\lambda)(9 - 6\lambda + \lambda^2 - 1) + 2(-6 + 2\lambda + 2) + 2(2 - 6 + 2\lambda) = -\lambda^3 + 12\lambda^2 - 36\lambda + 32$

Which is the characteristic polynomial of $\boldsymbol{A}$.

<u>Characteristic equation or secular equation:</u> The equation$|\boldsymbol{A} - \boldsymbol{\lambda I}| = 0$ is known as characteristic equation of matrix $\boldsymbol{A}$. So, here the required characteristic equation is $-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$.

<u>Eigen Values or Characteristic roots:</u> The roots of the characteristic equation$|\boldsymbol{A} - \boldsymbol{\lambda I}| = 0$ are known as Eigen values of $\boldsymbol{A}$.

Here $-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0 \Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0.$ $\therefore \lambda = 2,2,8$ are the Eigen values.

Eigen Vectors or Characteristic Vectors: Corresponding to each Eigen value λ, we have a non-zero vector $\boldsymbol{X}$ that satisfies the equation $(\boldsymbol{A}-\boldsymbol{\lambda I})X=0$.

The non-vector is known as the Eigen vector or characteristic vector or latent vector of $\boldsymbol{A}$ corresponds to the Eigen value λ. The Eigen vector is thus not unique, but of the form $\mu\boldsymbol{X}$, where μ is any scalar quantity.

Orthogonal Vectors: Two vectors $\boldsymbol{X}_1$ & $\boldsymbol{X}_2$ are called orthogonal, if $\boldsymbol{X}_1^\dagger\boldsymbol{X}_2=0$. Let, $\boldsymbol{X}_1=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ & $\boldsymbol{X}_2=\begin{bmatrix}x_4\\x_5\\x_6\end{bmatrix}$.

They are orthogonal if $x_1x_4+x_2x_5+x_3x_6=0$.

Normalized form of vector: The Eigen vector $\boldsymbol{X}$ of a matrix will be said normalized if $\boldsymbol{X}^\dagger\boldsymbol{X}=1$.

The normalized form of $\begin{bmatrix}a\\b\\c\end{bmatrix}$ is obtained by dividing $\sqrt{a^2+b^2+c^2}$. For instance the normalized form of $\begin{bmatrix}2\\3\\5\end{bmatrix}$ is $\frac{1}{\sqrt{38}}\begin{bmatrix}2\\3\\5\end{bmatrix}$.

Properties of Eigen values:

(i) The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal i.e. **trace of a matrix**.
(ii) The product of the Eigen values of a matrix $\boldsymbol{A}$ is equal to its determinant.
(iii) If λ is an Eigen value of a matrix $\boldsymbol{A}$, then $\frac{1}{\lambda}$ is the Eigen value of $\boldsymbol{A}^{-1}$.
(iv) If λ is an Eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its Eigen value.

Q33. The Eigen values of a real symmetric matrix are real.

⇒ Let λ be an Eigen value of the real symmetric matrix $\boldsymbol{A}$ corresponding to the Eigen vectors $\boldsymbol{X}$. $\therefore \boldsymbol{AX}=\lambda\boldsymbol{X}$….(i)

For the real symmetric matrix, $\boldsymbol{A}=\boldsymbol{A}^*$ (complex conjugate of $\boldsymbol{A}$) and $\boldsymbol{A}=\boldsymbol{A}^{\boldsymbol{T}}$ (transpose of $\boldsymbol{A}$) i.e. $\boldsymbol{A}^{\boldsymbol{T}}=\boldsymbol{A}^*$

Taking the complex conjugate of equation (i) we obtain, $\boldsymbol{A}^*\boldsymbol{X}^*=\lambda^*\boldsymbol{X}^* \Rightarrow \boldsymbol{AX}^*=\lambda^*\boldsymbol{X}^*$…..(ii) since $\boldsymbol{A}^*=\boldsymbol{A}$

Again taking transpose of equation (i), we get $(\boldsymbol{AX})^{\boldsymbol{T}}=(\lambda\boldsymbol{X})^{\boldsymbol{T}}$ $or,, \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}=\lambda\boldsymbol{X}^{\boldsymbol{T}}$ $\therefore \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{A}=\lambda\boldsymbol{X}^{\boldsymbol{T}}$….(iii) since $\boldsymbol{A}^{\boldsymbol{T}}=\boldsymbol{A}$

Now pre-multiplying equation (ii) by $\boldsymbol{X}^{\boldsymbol{T}}$ and post-multiplying equation (iii) by $\boldsymbol{X}^*$,

We obtain two equation as $\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{AX}^*=\lambda^*\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*$….(iv) [from equation(ii)] and $\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{AX}^*=\lambda\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*$….(v) [from equation(iii)]

Comparing equation (iv) & (v) we obtain, $\lambda^*\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*=\lambda\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*$ $or, (\lambda^*-\lambda)\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*=0$

Since $\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*\neq 0$, we have $(\lambda^*-\lambda)=0$ i.e. $\lambda^*=\lambda$. $\therefore \lambda=\lambda^*$. Which implies that the Eigen value is real.

Q34. The Eigen values of a real skew-symmetric matrix are purely imaginary.

⇒ Let λ be an Eigen value of the real skew-symmetric matrix $\boldsymbol{A}$ corresponding to the Eigen vectors $\boldsymbol{X}$. $\therefore \boldsymbol{AX}=\lambda\boldsymbol{X}$….(i)

For the real skew-symmetric matrix, $\boldsymbol{A}=\boldsymbol{A}^*$ (complex conjugate of $\boldsymbol{A}$) and $\boldsymbol{A}=-\boldsymbol{A}^{\boldsymbol{T}}$ (transpose of $\boldsymbol{A}$) i.e. $\boldsymbol{A}^{\boldsymbol{T}}=-\boldsymbol{A}^*$

Taking the complex conjugate of equation (i) we obtain, $\boldsymbol{A}^*\boldsymbol{X}^*=\lambda^*\boldsymbol{X}^* \Rightarrow \boldsymbol{AX}^*=\lambda^*\boldsymbol{X}^*$…..(ii) since $\boldsymbol{A}^*=\boldsymbol{A}$

Again taking transpose of equation (i), we get $(\boldsymbol{AX})^{\boldsymbol{T}}=(\lambda\boldsymbol{X})^{\boldsymbol{T}}$ $or,, \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}=\lambda\boldsymbol{X}^{\boldsymbol{T}}$ $\therefore \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{A}=-\lambda\boldsymbol{X}^{\boldsymbol{T}}$….(iii) since $\boldsymbol{A}^{\boldsymbol{T}}=-\boldsymbol{A}$

Now pre-multiplying equation (ii) by $\boldsymbol{X}^{\boldsymbol{T}}$ and post-multiplying equation (iii) by $\boldsymbol{X}^*$,

We obtain two equation as $\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{AX}^*=\lambda^*\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*$….(iv) [from equation(ii)] and $\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{AX}^*=-\lambda\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*$….(v) [from equation(iii)]

Comparing equation (iv) & (v) we obtain, $\lambda^*\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*=-\lambda\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*$ $or, (\lambda^*+\lambda)\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*=0$

Since $\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}^*\neq 0$, we have $(\lambda^*+\lambda)=0$ i.e. $\lambda^*=-\lambda$. $\therefore \lambda=-\lambda^*$. Which implies that the Eigen value is either **zero or imaginary**.

Q35. The Eigen values of a Hermitian matrix are real.

⇒ For a Hermitian matrix $\boldsymbol{A}$, $\boldsymbol{A}^\dagger=\boldsymbol{A}$, $\boldsymbol{A}^\dagger$ is the transposed conjugate of $\boldsymbol{A}$.

Let λ be an Eigen value of the Hermitian matrix $\boldsymbol{A}$ corresponding to the Eigen vectors $\boldsymbol{X}$. $\therefore \boldsymbol{AX}=\lambda\boldsymbol{X}$….(i)

Taking transposed conjugate of both sides we have, $\therefore (\boldsymbol{AX})^\dagger=(\lambda\boldsymbol{X})^\dagger$ $or, \boldsymbol{X}^\dagger\boldsymbol{A}^\dagger=\lambda^*\boldsymbol{X}^\dagger$ $\therefore \boldsymbol{X}^\dagger\boldsymbol{A}=\lambda^*\boldsymbol{X}^\dagger$….(ii) since $\boldsymbol{A}^\dagger=\boldsymbol{A}$

Now pre-multiplying equation (i) by $\boldsymbol{X}^\dagger$ and post-multiplying equation (ii) by $\boldsymbol{X}$,

We obtain two equation as $\boldsymbol{X}^\dagger\boldsymbol{AX}=\lambda\boldsymbol{X}^\dagger\boldsymbol{X}$….(iii) [from equation(i)] and $\boldsymbol{X}^\dagger\boldsymbol{AX}=\lambda^*\boldsymbol{X}^\dagger\boldsymbol{X}$….(iv) [from equation(ii)]

Comparing equation (iii) & (iv) we obtain, $\lambda\boldsymbol{X}^\dagger\boldsymbol{X}=\lambda^*\boldsymbol{X}^\dagger\boldsymbol{X}$ $or, (\lambda-\lambda^*)\boldsymbol{X}^\dagger\boldsymbol{X}=0$

Since $\boldsymbol{X}^\dagger\boldsymbol{X}\neq 0$, we have $(\lambda-\lambda^*)=0$ i.e. $\lambda=\lambda^*$. $\therefore \lambda=\lambda^*$.

[This means that the conjugate of λ is equal to itself.] This is only possible when λ is real. Thus the Eigen value of a Hermitian matrix are all real.

Q36. The Eigen values of a skew-Hermitian matrix are zero or purely imaginary.

Home work.

Q37. Any two Eigen vectors (corresponding to two distinct Eigen values of Hermitian matrix) are orthogonal.

⇒ Let $X_1, X_2$and are two Eigen vectors corresponding to two distinct Eigen valueλ_1, λ_2 respectively of a Hermitian matrix **A.**

$AX_1 = \lambda_1 X_1$……………(i) & $AX_2 = \lambda_2 X_2$……………(ii)

And $\boldsymbol{A} = (\boldsymbol{A}^*)^{\boldsymbol{T}} = \boldsymbol{A}^\dagger$; since **A** is Hermitian taking transpose conjugate of equation(i)

$(AX_1)^\dagger = (\lambda_1 X_1)^\dagger$ or, ${X_1}^\dagger A^\dagger = \lambda_1 {X_1}^\dagger$ or, ${X_1}^\dagger \mathrm{A} = \lambda_1 {X_1}^\dagger \ldots\ldots\ldots..(iii)$ Since $\lambda_1$is real for a Hermitian matrix.

Now, post multiplying equation (iii) byX_2 , we have ${X_1}^\dagger \mathrm{A}\, X_2 = \lambda_1 {X_1}^\dagger X_2$…….(iv)

Now, post multiplying equation (ii) by ${X_1}^\dagger$, we have ${X_1}^\dagger \mathrm{A}\, X_2 = \lambda_2 {X_1}^\dagger X_2$……...(v)

Comparing equation (iv) and (v) we get $\lambda_1 {X_1}^\dagger X_2 = \lambda_2 {X_1}^\dagger X_2$ or,$(\lambda_1 - \lambda_2){X_1}^\dagger X_2 = 0$

Since $\lambda_1 \neq \lambda_2$, we must have ${X_1}^\dagger X_2 = 0$

That means are X_1& X_2orthogonal to each other.

Q38. If $\boldsymbol{\lambda}$ is an Eigen value of a matrix $\boldsymbol{A}$, then $\frac{1}{\lambda}$ is the Eigen value of the matrix $\boldsymbol{A}^{-1}$ (if it exists).

⇒ Let λ be an Eigen value of matrix $\boldsymbol{A}$ corresponding to the Eigen vectors $\boldsymbol{X}$. ∴ $\boldsymbol{AX} = \lambda \boldsymbol{X}$….(i)

Now pre-multiplying equation (i) by $\boldsymbol{A}^{-1}$, we get $\boldsymbol{A}^{-1}\boldsymbol{AX} = \boldsymbol{A}^{-1}\boldsymbol{\lambda X}$ $or, \boldsymbol{IX} = \lambda(\boldsymbol{A}^{-1}\boldsymbol{X})$ $or, \boldsymbol{X} = \lambda \boldsymbol{A}^{-1}\boldsymbol{X}$ ∴ $\boldsymbol{A}^{-1}\boldsymbol{X} = \frac{1}{\lambda}\boldsymbol{X}$

The above equation shows that $\frac{1}{\lambda}$ is an Eigen value of matrix $\boldsymbol{A}^{-1}$ corresponding to the Eigen vector $\boldsymbol{X}$.

Q39. Eigen values of a matrix are invariant under a similarity transformation.

⇒ The matrices $\boldsymbol{A}$ & $\boldsymbol{B}$ are said to be related by a similarly transformation if $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP}$ or $\boldsymbol{A} = \boldsymbol{PBP}^{-1}$.

Let us consider two similar matrices $\boldsymbol{A}$ & $\boldsymbol{B}$ are related through similarity transformation such that, $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP}$.

The characteristic equation of the matrix $\boldsymbol{B}$ is $|\boldsymbol{B} - \lambda \boldsymbol{I}| = 0$ where λ is an Eigen value of $\boldsymbol{B}$.

i.e. $|\boldsymbol{P}^{-1}\boldsymbol{AP} - \lambda \boldsymbol{I}| = 0$

or, $|\boldsymbol{P}^{-1}\boldsymbol{AP} - \boldsymbol{P}^{-1}\lambda \boldsymbol{IP}| = 0$

or, $|\boldsymbol{P}^{-1}(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{P}| = 0$

or, $|\boldsymbol{P}^{-1}||(\boldsymbol{A} - \lambda \boldsymbol{I})||\boldsymbol{P}| = 0$

or, $|\boldsymbol{P}^{-1}||\boldsymbol{P}||(\boldsymbol{A} - \lambda \boldsymbol{I})| = 0$

or, $|\boldsymbol{P}^{-1}\boldsymbol{P}||(\boldsymbol{A} - \lambda \boldsymbol{I})| = 0$ ∴ $|(\boldsymbol{A} - \lambda \boldsymbol{I})| = 0$ This relation shows that λ is an Eigen value of $\boldsymbol{A}$ also. That proves the theorem.

Q40. Let $\boldsymbol{\lambda_1, \lambda_2, \ldots, \lambda_n}$ be the Eigen values of $\boldsymbol{A}$ (are not necessary to be distinct). Then, $\sum_{i=1}^{n} \boldsymbol{\lambda_i} = \boldsymbol{tr}(\boldsymbol{A})$ and $\prod_{i=1}^{n} \boldsymbol{\lambda_i} = \boldsymbol{det}(\boldsymbol{A}) = |\boldsymbol{A}|$.

⇒ $f(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$.

Thus $f(0) = \det(-\boldsymbol{A}) = (-1)^n \det(\boldsymbol{A}) = (0 - \lambda_1)(0 - \lambda_2) \ldots (0 - \lambda_n) = (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n = (-1)^n \prod_{i=1}^{n} \lambda_i$

Therefore, $det(A) = \prod_{i=1}^{n} \lambda_i$

Also, by diagonal expansion on the following determinant, $f(\lambda) = \begin{vmatrix} -a_{11} + \lambda & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & -a_{22} + \lambda & \ldots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \ldots & -a_{nn} + \lambda \end{vmatrix} = \lambda^n - (\sum_{i=1}^{n} a_{ii})\lambda^{n-1} + \cdots + (-1)^n \prod_{i=1}^{n} \lambda_i$

And by the expansion of $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n) = \lambda^n - (\sum_{i=1}^{n} a_{ii})\lambda^{n-1} + \cdots + (-1)^n \prod_{i=1}^{n} \lambda_i$

Therefore $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = tr(\boldsymbol{A})$.

Example: $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} = [a_{ii}]_{3\times3}$.

The Eigen values of $\boldsymbol{A}$ are $\lambda = 1{,}1$ & 6. Then $\lambda_1 + \lambda_2 + \lambda_3 = 1 + 1 + 6 = 8$ & $tr(\boldsymbol{A}) = a_{11} + a_{22} + a_{33} = 0 + 3 + 5 = 8$

Also $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1 \cdot 1 \cdot 6 = 6$ & $\det(\boldsymbol{A}) = 16 - 10 = 6$

Q41. Let $\boldsymbol{u}$ be the Eigen vector of $\boldsymbol{A}_{n\times n}$ associated with Eigen value $\boldsymbol{\lambda}$. Then, the Eigenvalue of $\boldsymbol{a_k A^k + a_{k-1} A^{k-1} + \cdots + a_1 A + a_0 I}$ associated with the Eigen vector $\boldsymbol{u}$ is, $\boldsymbol{a_k \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0}$, $\boldsymbol{a_k, a_{k-1}, \ldots, a_1, a_0}$ are real numbers and $\boldsymbol{k}$ is a positive integer.

$\Rightarrow \left(a_k \boldsymbol{A}^{\boldsymbol{k}} + a_{k-1}\boldsymbol{A}^{\boldsymbol{k-1}} + \cdots + a_1\boldsymbol{A} + a_0\boldsymbol{I}\right)u$

$= a_k \boldsymbol{A}^{\boldsymbol{k}} u + a_{k-1}\boldsymbol{A}^{\boldsymbol{k-1}} u + \cdots + a_1 \boldsymbol{A} u + a_0 \boldsymbol{I} u = a_k \boldsymbol{\lambda}^{\boldsymbol{k}} u + a_{k-1}\boldsymbol{\lambda}^{\boldsymbol{k-1}} u + \cdots + a_1 \boldsymbol{\lambda} u + a_0 u = \left(a_k \boldsymbol{\lambda}^{\boldsymbol{k}} + a_{k-1}\boldsymbol{\lambda}^{\boldsymbol{k-1}} + \cdots + a_1\boldsymbol{\lambda} + a_0\right)u$

Since $\boldsymbol{A}^{\boldsymbol{j}} u = \boldsymbol{A}^{\boldsymbol{j-1}}(\boldsymbol{A}u) = \boldsymbol{A}^{\boldsymbol{j-1}}\lambda u = \lambda \boldsymbol{A}^{\boldsymbol{j-1}} u = \lambda \boldsymbol{A}^{\boldsymbol{j-2}}(\boldsymbol{A}u) = \lambda^2 \boldsymbol{A}^{\boldsymbol{j-2}} u = \cdots = \lambda^{j-1}\boldsymbol{A}u = \lambda^j u$.

Example: $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$, what is the Eigenvalues of $2\boldsymbol{A}^{\boldsymbol{100}} + 4\boldsymbol{A} - 12\boldsymbol{I}$.

$\Rightarrow$ The Eigen values of $\boldsymbol{A}$ are -5 and 7. Thus, the Eigen values of the $2\boldsymbol{A}^{\boldsymbol{100}} + 4\boldsymbol{A} - 12\boldsymbol{I}$,

$2(-5)^{\boldsymbol{100}} + 4(-5) - 12 = 2 \cdot 5^{100} - 32 =$ and $2(7)^{\boldsymbol{100}} + 4 \cdot 7 - 12 = 2 \cdot 7^{100} - 16 =$

Example: Let λ be the Eigen value of $\boldsymbol{A}$. Then, we denote $e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \cdots + \frac{\boldsymbol{A}^n}{n!} + \cdots = \frac{\sum_{i=0}^{\infty}\boldsymbol{A}^i}{i!}$.

$\Rightarrow$ Then $e^{\boldsymbol{A}}$ has Eigen values, $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots + \frac{\lambda^n}{n!} + \cdots = \frac{\sum_{i=0}^{\infty}\lambda^i}{i!}$

Q42.. Find the Eigen values and Eigen vectors of matrix $\begin{pmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} \end{pmatrix}$. Is this matrix is unitary?

Ans: Here matrix A $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Characteristic matrix $A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{pmatrix}$

Characteristic polynomial is $det(A - \lambda I)$ or, $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = (1-\lambda)(\lambda+1)(\lambda-1)$

Then characteristic equation $det(A - \lambda I) = 0$ or, $(1-\lambda)(\lambda+1)(\lambda-1) = 0$

Then Eigen values are $\mathbf{1}, -\mathbf{1}, \mathbf{1}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **1.**

Then $\begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 0-1 & 1 \\ 0 & 1 & 0-1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

That means $-x_{21} + x_{31} = 0$ and $x_{21} - x_{31} = 0$

Hence $x_{21} = x_{31}(say\ a)$ and $x_{11}$is in determined (say b) any arbitrary value.

Then $\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} b \\ a \\ a \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value $-\mathbf{1}$.

Then $\begin{pmatrix} 1+1 & 0 & 0 \\ 0 & 0+1 & 1 \\ 0 & 1 & 0+1 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

That means $2x_{12} = 0$, $x_{22} + x_{32} = 0$ and $x_{22} + x_{32} = 0$

Hence $x_{22} = -x_{32}(say\ a')$ and $x_{12} = 0$

Then $\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ a' \\ -a' \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_3 = \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **1.**

Then $\begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 0-1 & 1 \\ 0 & 1 & 0-1 \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

That means $-x_{23} + x_{33} = 0$ and $x_{23} - x_{33} = 0$

Hence $x_{23} = x_{33}(say\ a'')$ and $x_{13}$is in determined $(say\ b'')$ any arbitrary value.

Then $\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} b'' \\ a'' \\ a'' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence eigen vectors are $\begin{pmatrix} b \\ a \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ a' \\ -a' \end{pmatrix}, \begin{pmatrix} b'' \\ a'' \\ a'' \end{pmatrix}$ we choose orthogonal set as $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

As $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Then $A^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Then $A^\dagger = (A^*)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Then $AA^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

Hence A is an unitary matrix.

Q43.. Find the Eigen values and Eigen vectors of matrix $\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$**.**

Ans: Here matrix A $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Characteristic matrix $A - \lambda I = \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix}$.

Characteristic polynomial is $det(A - \lambda I)$ or, $\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{\lambda - 2 - 1\} - 1\{1 + 2 - \lambda\}$

Then characteristic equation $det(A - \lambda I) = 0$ or, $(2-\lambda)\{4 - 4\lambda + \lambda^2 - 1\} + 2\{\lambda - 3\} = 0$

Or, $(2-\lambda)\{\lambda^2 - 4\lambda + 3\} + 2\{\lambda - 3\} = 0$ or, $(2-\lambda)\{\lambda^2 - 3\lambda - \lambda + 3\} + 2\{\lambda - 3\} = 0$ or, $(2-\lambda)\{(\lambda - 3)(\lambda - 1)\} + 2\{\lambda - 3\} = 0$

Or, $(\lambda - 3)\{(2-\lambda)(\lambda - 1) + 2\} = 0$ or, $(\lambda - 3)\{2\lambda - 2 - \lambda^2 + \lambda + 2\} = 0$ or, $(\lambda - 3)(-\lambda)(\lambda - 3) = 0$

Then Eigen values are $\mathbf{0}, \mathbf{3}, \mathbf{3}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **0.**

Then$\begin{pmatrix} 2-0 & -1 & -1 \\ -1 & 2-0 & -1 \\ -1 & -1 & 2-0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Or, $2x_{11} - x_{21} - x_{31} = 0$ and $-x_{11} + 2x_{21} - x_{31} = 0$ and $-x_{11} - x_{21} + 2x_{31} = 0$

Hence $x_{11} = \frac{(x_{21}+x_{31})}{2}$, $x_{21} = \frac{(x_{11}+x_{31})}{2}$ and $x_{31} = \frac{(x_{11}+x_{21})}{2}$ then $x_{11} = x_{21} = x_{31}$=$(say\ a)$

Then $\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value 3

$\begin{pmatrix} 2-3 & -1 & -1 \\ -1 & 2-3 & -1 \\ -1 & -1 & 2-3 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Then $-x_{12}-x_{22}-x_{32}=0$ then, $x_{12}=-(x_{22}+x_{32})$ hence x_{22} and x_{32} are arbitrary (say a′ and b′)

Then $\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}=\begin{pmatrix} -(a'+b') \\ a' \\ b' \end{pmatrix}$ Hence infinite number of solution are possible.

Similarly Eigen vector of A belonging to the Eigen value 3 $\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}=\begin{pmatrix} -(c'+d') \\ c' \\ d' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence Eigen vectors are $\begin{pmatrix} a \\ a \\ a \end{pmatrix},\begin{pmatrix} -(a'+b') \\ a' \\ b' \end{pmatrix},\begin{pmatrix} -(c'+d') \\ c' \\ d' \end{pmatrix}$ we choose orthogonal set as $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Note that orthonormal Eigen vectors are $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},\frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Q44. Find the Eigen values and a set of orthogonal Eigen vectors of matrix $\begin{pmatrix} \mathbf{4} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{4} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{4} \end{pmatrix}$.

ANS: Here matrix A $=\begin{pmatrix} \mathbf{4} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{4} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{4} \end{pmatrix}$. Characteristic matrix $A-\lambda I=\begin{pmatrix} \mathbf{4-\lambda} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{4-\lambda} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{4-\lambda} \end{pmatrix}$.

Characteristic polynomial is $det(A-\lambda I)$ or, $\begin{vmatrix} \mathbf{4-\lambda} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{4-\lambda} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{4-\lambda} \end{vmatrix}=(4-\lambda)\{(4-\lambda)^2-1\}+1\{\lambda-4-1\}-1\{1+4-\lambda\}$

Then characteristic equation $det(A-\lambda I)=0$ or, $(4-\lambda)\{16-8\lambda+\lambda^2-1\}+2\{\lambda-5\}=0$

Or, $(4-\lambda)\{\lambda^2-8\lambda+15\}+2\{\lambda-5\}=0$ or, $(4-\lambda)\{\lambda^2-5\lambda-3\lambda+15\}+2\{\lambda-5\}=0$ or, $(4-\lambda)\{(\lambda-5)(\lambda-3)\}+2\{\lambda-5\}=0$

Or, $(\lambda-5)\{(4-\lambda)(\lambda-3)+2\}=0$ or, $(\lambda-5)\{4\lambda-12-\lambda^2+3\lambda+2\}=0$ or, $(\lambda-5)\{7\lambda-10-\lambda^2\}=0$

or, $(\lambda-5)\{5\lambda+2\lambda-10-\lambda^2\}=0$ 0R, $-(\lambda-5)(\lambda-5)(\lambda-2)=0$. Hence the Eigen values are $\mathbf{5},\mathbf{5},\mathbf{2}$

Let $X_1=\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **5.**

Then $\begin{pmatrix} \mathbf{4-5} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{4-5} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{4-5} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $or,\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Then $-x_{11}-x_{21}-x_{31}=0$ hence $x_{11}=-(x_{21}+x_{31})$ hence x_{21} and x_{31} are arbitrary (say a and b)

Then $\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}=\begin{pmatrix} -(a+b) \\ a \\ b \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2=\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value 5

Similarly Then $\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}=\begin{pmatrix} -(a'+b') \\ a' \\ b' \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_3=\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **2.**

Then $\begin{pmatrix} \mathbf{4-2} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{4-2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{4-2} \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $or,\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Or, $2x_{13}-x_{23}-x_{33}=0$ and $-x_{13}+2x_{23}-x_{33}=0$ and $-x_{13}-x_{23}+2x_{33}=0$

Hence $x_{13}=\frac{(x_{23}+x_{33})}{2}$, $x_{23}=\frac{(x_{13}+x_{33})}{2}$ and $x_{33}=\frac{(x_{13}+x_{23})}{2}$ then $x_{13}=x_{23}=x_{33}=(say\ a'')$

Then $\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} a'' \\ a'' \\ a'' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence Eigen vectors are $\begin{pmatrix} -(a+b) \\ a \\ b \end{pmatrix}, \begin{pmatrix} -(a'+b') \\ a' \\ b' \end{pmatrix}, \begin{pmatrix} a \\ a \\ a \end{pmatrix}$, we choose orthogonal set as $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Q45. Find the Eigen values and Eigen vectors of matrix $\begin{pmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} \end{pmatrix}$.

Ans: Here the matrix is A=$\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$. Then the Characteristic matrix $A - \lambda I = \begin{pmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix}$

Characteristic polynomial is $det(A - \lambda I)$ or, $\begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = 10 - 7\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6$

Then characteristic equation $det(A - \lambda I) = 0$ or, $\lambda^2 - 7\lambda + 6 = 0$ or, $\lambda^2 - 6\lambda - \lambda + 6 = 0$ or, $(\lambda - 6)(\lambda - 1) = 0$

Hence the Eigen values are $\mathbf{6}, \mathbf{1}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **6.**

Then $\begin{pmatrix} 5-6 & 2 \\ 2 & 2-6 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $-x_{11} + 2x_{21}$=0 and $2x_{11} - 4x_{21}$=0 that means $x_{11} = 2x_{21}$= (say 2a)

Then $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 2a \\ a \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **1.**

Then $\begin{pmatrix} 5-1 & 2 \\ 2 & 2-1 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $4x_{12}$+$2x_{22} = 0$ and $2x_{12}$+$1x_{22} = 0$ that means $x_{12} = -\frac{x_{22}}{2}$ = (say a')

Then $\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} a' \\ -2a' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence Eigen vectors are $\begin{pmatrix} 2a \\ a \end{pmatrix}, \begin{pmatrix} a' \\ -2a' \end{pmatrix}$ we choose orthogonal set as $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ Note that orthonormal Eigen vectors are $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

<u>Q46 Find the Eigen values and a set of orthogonal Eigen vectors of matrix</u> $\begin{pmatrix} \mathbf{10} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{6} & \mathbf{-4} \\ \mathbf{0} & \mathbf{-4} & \mathbf{6} \end{pmatrix}$**.**

Ans: Here the matrix is A=$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$. Then the Characteristic matrix $A - \lambda I = \begin{pmatrix} 10-\lambda & 0 & 0 \\ 0 & 6-\lambda & -4 \\ 0 & -4 & 6-\lambda \end{pmatrix}$

Characteristic polynomial is $det(A - \lambda I)$ or, $\begin{vmatrix} 10-\lambda & 0 & 0 \\ 0 & 6-\lambda & -4 \\ 0 & -4 & 6-\lambda \end{vmatrix} = (10-\lambda)\{(6-\lambda)^2 - 16\} = (10-\lambda)\{36 + \lambda^2 - 12\lambda - 16\}$

Then characteristic equation $det(A - \lambda I) = 0$ or, $(10-\lambda)\{\lambda^2 - 12\lambda + 20\} = 0$ or, $(10-\lambda)\{\lambda^2 - 10\lambda - 2\lambda + 20\} = 0$ or,

Or, $(10-\lambda)\{(\lambda - 10)(\lambda - 2)\} = 0$. Hence the Eigen values are $\mathbf{10}, \mathbf{2}, \mathbf{10}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **10.**

Then $\begin{pmatrix} 10-10 & 0 & 0 \\ 0 & 6-10 & -4 \\ 0 & -4 & 6-10 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Then $-4x_{21} - 4x_{31} = 0$ & $-4x_{21} - 4x_{31} = 0$

Hence $x_{21} = -x_{31}(say\ a)$ and $x_{11}$is in determined (say b) any arbitrary value.

Then $\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} b \\ a \\ -a \end{pmatrix}$ Hence infinite number of solution are possible.

Similarly Eigen vector of A belonging to the Eigen value 10 $\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} b' \\ a' \\ -a' \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value 2

Then $\begin{pmatrix} 10-2 & 0 & 0 \\ 0 & 6-2 & -4 \\ 0 & -4 & 6-2 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Then $8x_{12} = 0$ and $4x_{22} - 4x_{32} = 0$ and $-4x_{22} + 4x_{32} = 0$.

That means $x_{12} = 0$ and $x_{22} = x_{32}(say\ a'')$ Then $\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ a'' \\ a'' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence Eigen vectors are $\begin{pmatrix} b \\ a \\ -a \end{pmatrix}, \begin{pmatrix} 0 \\ a'' \\ a'' \end{pmatrix}, \begin{pmatrix} b' \\ a' \\ -a' \end{pmatrix}$ we choose orthogonal set as $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ orthonormal Eigen vectors are $\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

<u>Q47 Find the Eigen values and Eigen vectors of matrix</u> $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{1} \end{pmatrix}$

Ans: Here the matrix is A= $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. Hence the Characteristic matrix $A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}$

Characteristic polynomial is $det(A - \lambda I)$ or, $\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$

Then characteristic equation $det(A - \lambda I) = 0$ or, $\lambda^2 - 2\lambda - 3 = 0$ or, $\lambda^2 - 3\lambda + \lambda - 3 = 0$ or, $(\lambda - 3)(\lambda + 1) = 0$

Hence the Eigen values are $\mathbf{3}, -\mathbf{1}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **3.**

Then $\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $-2x_{11} + x_{21}$=0 and $4x_{11} - -2x_{21}$=0 that means $x_{11} = \frac{x_{21}}{2}$ = (say a).

Then $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} a \\ 2a \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value $-$**1.**

Then $\begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $2x_{12} + x_{22} = 0$ and $4x_{12} + 2x_{22} = 0$ that means $x_{12} = -\frac{x_{22}}{2}$ = (say a')

Then $\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} a' \\ -2a' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence Eigen vectors are $\begin{pmatrix} a \\ 2a \end{pmatrix}, \begin{pmatrix} a' \\ -2a' \end{pmatrix}$ we chose one set of them as $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ **<u>note that they are not orthogonal set</u>**

Q48. Find the Eigen values and Eigen vectors of matrix $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \end{pmatrix}$

Ans: Here the matrix is A= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$. Hence the Characteristic matrix $A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 2 \\ 0 & 2 & 0-\lambda \end{pmatrix}$

Characteristic polynomial is $det(A - \lambda I)$ or, $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 2 \\ 0 & 2 & 0-\lambda \end{vmatrix} = (1-\lambda)\{\lambda^2 - 4\} = (1-\lambda)(\lambda+2)(\lambda-2)$

Then characteristic equation $det(A - \lambda I) = 0$ or, $(1-\lambda)(\lambda+2)(\lambda-2) = 0$ Hence the Eigen values are $\mathbf{1}, \mathbf{-2}, \mathbf{2}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **1.**

Then $\begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 0-1 & 2 \\ 0 & 2 & 0-1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Then $-x_{21} + 2x_{31} = 0$ and $2x_{21} - x_{31} = 0$ only possible solution of these equation is $x_{21} = 0$ $x_{31} = 0$ and $x_{11}$is in determined (say a)

Then $\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value -2

Then $\begin{pmatrix} 1-(-2) & 0 & 0 \\ 0 & 0-(-2) & 2 \\ 0 & 2 & 0-(-2) \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Hence $3x_{12} = 0$ and $2x_{22}+2x_{32} = 0$ and $2x_{22}+2x_{32} = 0$ then $x_{22} = -x_{32} = (say\ a')$ and $x_{12} = 0$

Then $\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ a' \\ -a' \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_3 = \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **2.**

Then $\begin{pmatrix} 1-2 & 0 & 0 \\ 0 & 0-2 & 2 \\ 0 & 2 & 0-2 \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Then $-x_{13} = 0\ and - 2x_{23}+2x_{33} = 0$ and $2x_{23} - 2x_{33} = 0$ then $x_{23} = x_{33} = (say\ a'')$ and $x_{13} = 0$

Then $\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ a'' \\ a'' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence Eigen vectors are $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a' \\ -a' \end{pmatrix}, \begin{pmatrix} 0 \\ a'' \\ a'' \end{pmatrix}$ we choose orthogonal set as $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Note that orthonormal Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Q49.Find the Eigen values and normalized Eigen vectors of matrix$\begin{pmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{2} \end{pmatrix}$

Ans: Here the matrix is A=$\begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}$. Then the Characteristic matrix $det(A - \lambda I)$ =$\begin{pmatrix} 1-\lambda & -2 \\ -2 & 2-\lambda \end{pmatrix}$

Characteristic polynomial is $det(A - \lambda I)$ or,$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 4 = 2 - 3\lambda + \lambda^2 - 4 = \lambda^2 - 3\lambda - 2$

Then characteristic equation $det(A - \lambda I) = 0$ or, $\lambda^2 - 3\lambda - 2 = 0$ then λ= $\frac{3\pm\sqrt{17}}{2}$

Then Eigen values are $\frac{\mathbf{3+\sqrt{17}}}{\mathbf{2}}, \frac{\mathbf{3-\sqrt{17}}}{\mathbf{2}}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value. $\frac{\mathbf{3+\sqrt{17}}}{\mathbf{2}}$

Then $\begin{pmatrix} 1 - \frac{\mathbf{3+\sqrt{17}}}{\mathbf{2}} & -2 \\ -2 & 2 - \frac{\mathbf{3+\sqrt{17}}}{\mathbf{2}} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} \frac{\mathbf{-1-\sqrt{17}}}{\mathbf{2}} & -2 \\ -2 & \frac{\mathbf{1-\sqrt{17}}}{\mathbf{2}} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then, $\frac{\mathbf{-1-\sqrt{17}}}{\mathbf{2}}x_{11} - 2x_{21} = 0$ and $-2x_{11} + \frac{\mathbf{1-\sqrt{17}}}{\mathbf{2}}x_{21} = 0$ or, $(\mathbf{-1-\sqrt{17}})x_{11} - 4x_{21} = 0$ …..(i)and $-4x_{11} + (\mathbf{1-\sqrt{17}})x_{21} = 0$…….(i)

Now from (i) $(\mathbf{1-\sqrt{17}})(\mathbf{-1-\sqrt{17}})x_{11} - (\mathbf{1-\sqrt{17}})4x_{21} = 0$ or 16 $x_{11} - (\mathbf{1-\sqrt{17}})4x_{21} = 0$ or, $4x_{11} - (\mathbf{1-\sqrt{17}})x_{21} = 0$……..(iii)

Then x_{11}=$\frac{(\mathbf{1-\sqrt{17}})x_{21}}{4}$ then $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} (\mathbf{1-\sqrt{17}}) \\ 4 \end{pmatrix}$ say .

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value $\frac{\mathbf{3-\sqrt{17}}}{\mathbf{2}}$

Then $\begin{pmatrix} 1 - \frac{\mathbf{3-\sqrt{17}}}{\mathbf{2}} & -2 \\ -2 & 2 - \frac{\mathbf{3-\sqrt{17}}}{\mathbf{2}} \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} \frac{\mathbf{-1+\sqrt{17}}}{\mathbf{2}} & -2 \\ -2 & \frac{\mathbf{1+\sqrt{17}}}{\mathbf{2}} \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then, $\frac{\mathbf{-1+\sqrt{17}}}{\mathbf{2}}x_{12} - 2x_{22} = 0$ or, $(\mathbf{-1+\sqrt{17}})x_{12} - 4x_{22} = 0$ …….(i) and $-2\ x_{12} + \frac{\mathbf{1+\sqrt{17}}}{\mathbf{2}}x_{22} = 0$ $or, -4x_{12} + (\ \mathbf{1+\ \sqrt{17}})\ x_{22} = 0$…..(ii)

From (i) we get $(\ +\sqrt{17})(\mathbf{-1+\sqrt{17}})x_{12} - (\ \mathbf{1+\ \sqrt{17}})\mathbf{4}x_{22} = 0$ or, $16x_{12} - 4\ (\ \mathbf{1+\ \sqrt{17}})\ x_{22} = 0$ or, $4x_{12} - (\ \mathbf{1+\ \sqrt{17}})\ x_{22} = 0$…..(iii)

Hence $x_{12} = -\ \frac{(\ \mathbf{1+\sqrt{17}})\ x_{22}}{4}$ then$\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{1+\sqrt{17}}) \\ -4 \end{pmatrix}$ say

Then normalized eigen functions are $\frac{1}{\sqrt{((\mathbf{1-\sqrt{17}}))^2 + 16}}\begin{pmatrix} (\mathbf{1-\sqrt{17}}) \\ 4 \end{pmatrix}, \frac{1}{\sqrt{((\mathbf{1+\sqrt{17}}))^2 + 16}}\begin{pmatrix} (\mathbf{1+\sqrt{17}}) \\ -4 \end{pmatrix}$

Q50.Find the Eigen values and normalized Eigen vectors of matrix$\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$

Here the matrix is A=$\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$Then the Characteristic matrix $det(A - \lambda I)$ =$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$

Characteristic polynomial is $det(A - \lambda I)$ or,$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$=$(1-\lambda)^2 - 1 = -2\lambda + \lambda^2 = \lambda(\lambda - 2)$

Then characteristic equation $det(A - \lambda I) = 0$ or,$\lambda(\lambda - 2) = 0$ then λ= 0,2 Hence the Eigen values are $\mathbf{0}, \mathbf{2}$

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **0.**

Then$\begin{pmatrix} 1-0 & 1 \\ 1 & 1-0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$or, $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $x_{11} + x_{21}$=0 and $x_{11} = -x_{21}$= (say a).

Then $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix}$ Hence infinite number of solution are possible.

Let $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ be the Eigen vector of A belonging to the Eigen value **2.**

Then $\begin{pmatrix} 1-2 & 1 \\ 1 & 1-2 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or, $\begin{pmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $-x_{12}+x_{22} = 0$ and $x_{12}-x_{22} = 0$ that means $x_{12} = x_{22} =$ (say a')

Then $\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} a' \\ a' \end{pmatrix}$ Hence infinite number of solution are possible.

Hence Eigen vectors are $\begin{pmatrix} a \\ -a \end{pmatrix}, \begin{pmatrix} a' \\ a' \end{pmatrix}$ we chose one set of them as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Cayley-Hamilton Theorem: According to the Cayley-Hamilton theorem, every square matrix $\boldsymbol{A}$ satisfies its own characteristic equation. Let the characteristic **polynomial** of $\boldsymbol{A}$ be $\chi(\lambda) = det\,[\boldsymbol{A} - \lambda \boldsymbol{I}]$.

When the determinant is fully expanded and terms in the same power of λ are collected, one obtains $f(\lambda) = \sum_{j=0}^{n} f_j \lambda^j$

Q36. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

⇒ The characteristic equation of $\boldsymbol{A}$ is, $\begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad or,\ \lambda^2 - 4\lambda - 5 = 0$….(i)

By Cayley-Hamilton theorem, $\boldsymbol{A}$ must satisfy its characteristic equation (i), so that $\boldsymbol{A}^2 - 4\boldsymbol{A} - 5\boldsymbol{I} = 0$….(ii)

Now $\boldsymbol{A}^2 - 4\boldsymbol{A} - 5\boldsymbol{I} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$ [This verifies the theorem]

Multiplying (ii) by $\boldsymbol{A}^{-1}$, we get $\boldsymbol{A} - 4\boldsymbol{I} - 5\boldsymbol{A}^{-1} = 0 \quad or, \quad \boldsymbol{A}^{-1} = \frac{1}{5}(\boldsymbol{A} - 4\boldsymbol{I}) = \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

Now dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$ by the polynomial $\lambda^2 - 4\lambda - 5$, we obtain

$\lambda^5 - 4\lambda^4 - 7\lambda^3 - \lambda - 10I = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5 = \lambda + 5$

Hence $\boldsymbol{A}^5 - 4\boldsymbol{A}^4 - 7\boldsymbol{A}^3 + 11\boldsymbol{A}^2 - \boldsymbol{A} - 10\boldsymbol{I} = \boldsymbol{A} + 5\boldsymbol{I}$, which is a linear polynomial in $\boldsymbol{A}$.

Q51. Find the characteristic equation of the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{-4} & \mathbf{-4} \end{bmatrix}$ and hence find its inverse.

⇒ The characteristic equation of $\boldsymbol{A}$ is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 20\lambda + 8 = 0$

By Cayley-Hamilton theorem, $\boldsymbol{A}^3 - 20\boldsymbol{A} + 8\boldsymbol{I} = 0$, whence $\boldsymbol{A}^{-1} = \frac{5}{2}\boldsymbol{I} - \frac{1}{8}\boldsymbol{A}^2 = \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 30 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

Q52. Find the characteristic equation of the matrix $A = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{bmatrix}$ and hence find its inverse. Also find the matrix represented by

$\boldsymbol{A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.}$

⇒ The characteristic equation of $\boldsymbol{A}$ is $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem, $\boldsymbol{A}^3 - 5\boldsymbol{A}^2 + 7\boldsymbol{A} - 3\boldsymbol{I} = 0$……(i)

Multiplying (ii) by $\boldsymbol{A}^{-1}$, we get $\boldsymbol{A}^2 - 5\boldsymbol{A} + 7\boldsymbol{I} - 3 = 0 \quad or,\ \boldsymbol{A}^{-1} = \frac{1}{3}[\boldsymbol{A}^2 - 5\boldsymbol{A} + 7\boldsymbol{I}]$…..(ii)

But $\boldsymbol{A}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$\therefore \boldsymbol{A}^2 - 5\boldsymbol{A} + 7\boldsymbol{I} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 3 \\ -1 & -1 & 2 \end{bmatrix}$

Hence from (ii), $\boldsymbol{A^{-1}} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 3 \\ -1 & -1 & 2 \end{bmatrix}$

$\boldsymbol{A^8} - 5\boldsymbol{A^7} + 7\boldsymbol{A^6} - 3\boldsymbol{A^5} + \boldsymbol{A^4} - 5\boldsymbol{A^3} + 8\boldsymbol{A^2} - 2\boldsymbol{A} + \boldsymbol{I}$

$= \boldsymbol{A^5}(\boldsymbol{A}^3 - 5\boldsymbol{A^2} + 7\boldsymbol{A} - 3\boldsymbol{I}) + \boldsymbol{A}(\boldsymbol{A}^3 - 5\boldsymbol{A^2} + 7\boldsymbol{A} - 3\boldsymbol{I}) + \boldsymbol{A^2} + \boldsymbol{A} + \boldsymbol{I} = \boldsymbol{A^2} + \boldsymbol{A} + \boldsymbol{I}$ $[\because \boldsymbol{A^2} + \boldsymbol{A} + \boldsymbol{I} = 0]$

$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

Reduction to Diagonal Form

If a square matrix $\boldsymbol{A}$ of order n has n linearly independent Eigen vectors, than a matrix $\boldsymbol{P}$ can be found such that $\boldsymbol{P^{-1}AP}$ is a diagonal matrix.

[This result will be proved for a square matrix of order 3 but the method will be capable of easy extension to matrices of any order]

Let $\boldsymbol{A}$ be a matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its Eigen values and $\boldsymbol{X_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \boldsymbol{X_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *and* $\boldsymbol{X_3} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding Eigen vectors.

Denoting the square matrix $[X_1 X_2 X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ by $\boldsymbol{P}$, we have

$\boldsymbol{AP} = \boldsymbol{A}[X_1 X_2 X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3] = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \boldsymbol{PD}$ [$\boldsymbol{D}$ is the diagonal matrix]

$\therefore \boldsymbol{P^{-1}AP} = \boldsymbol{P^{-1}PD} = \boldsymbol{D}$, which proves the theorem.

Similarity of matrices: A square matrix $\widehat{\boldsymbol{A}}$ of order n is called similar to a square matrix $\boldsymbol{A}$ of order n if $\widehat{\boldsymbol{A}} = \boldsymbol{P^{-1}AP}$ for some non-singular $n \times n$ matrix $\boldsymbol{P}$. This transformation of a matrix $\boldsymbol{A}$ by a non-singular matrix $\boldsymbol{P}$ to $\widehat{\boldsymbol{A}}$ called a similarity transformation.

If $\boldsymbol{x}$ is an Eigen vector of $\boldsymbol{A}$, then $y = P^{-1}\boldsymbol{x}$ is an Eigen vector of $\widehat{\boldsymbol{A}}$ corresponding to the same value.

Powers of a Matrix: Diagonalisation of a matrix is quite useful for obtaining powers of a matrix. Let $\boldsymbol{A}$ be the square matrix. Then a non-singular matrix $\boldsymbol{P}$ can be found such that,

$\boldsymbol{D} = \boldsymbol{P^{-1}AP} \therefore \boldsymbol{D}^2 = (\boldsymbol{P^{-1}AP})(\boldsymbol{P^{-1}AP}) = \boldsymbol{P^{-1}A^2P}$ $[\because \boldsymbol{P^{-1}P} = 1]$

Similarly $\boldsymbol{D}^3 = \boldsymbol{P^{-1}A^3P}$ and in general $\boldsymbol{D}^n = \boldsymbol{P^{-1}A^nP}$….(i)

To obtain $\boldsymbol{A^n}$**,** pre-multiply (i) by $\boldsymbol{P}$ and post-multiply by $\boldsymbol{P^{-1}}$. Then $\boldsymbol{PD}^n\boldsymbol{P}^{-1} = \boldsymbol{PP}^{-1}\boldsymbol{A}^n\boldsymbol{PP}^{-1} = \boldsymbol{A}^n$ which gives $\boldsymbol{A}^n$.

Thus, $\boldsymbol{A}^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}$ where $\boldsymbol{D}^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$

Working procedure: (i) Find the Eigen values of the square matrix $\boldsymbol{A}$.

(ii) Find the corresponding Eigen vectors and write the model matrix $\boldsymbol{P}$.

(iii) Find the diagonal matrix $\boldsymbol{D}$ from $\boldsymbol{D} = \boldsymbol{P^{-1}AP}$.

(iv) Obtain $\boldsymbol{A}^n$ from $\boldsymbol{A}^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}$.

Q53. Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form.

⇒ The characteristic equation of A is $\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0 \quad or, \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$

Solving we get $\lambda_1 = 1, \lambda_2 = \sqrt{5}, \lambda_3 = -\sqrt{5}$ as the Eigen vector is given by, $-2x + 2y - 2z = 0, x + y + z = 0, -x - y - z = 0$

Solving the first two equation, we get $\frac{x}{2} = \frac{y}{0} = \frac{z}{-2}$ giving the Eigen vector $(1,0,-1)$

When $\lambda = \sqrt{5}$, the corresponding Eigen vector is given by, $(-1-\sqrt{5})x + 2y - 2z = 0,\ x + (2-\sqrt{5})y + z = 0,\ -x - y - \sqrt{5}z = 0$

Solving 2nd and 3rd equations, we get $\frac{x}{6-2\sqrt{5}} = \frac{y}{-1+\sqrt{5}} = \frac{z}{1-\sqrt{5}} \quad or, \frac{x}{\sqrt{5}-1} = \frac{y}{1} = \frac{z}{-1}$ giving the Eigen vector $(\sqrt{5}-1, -1, 1)$

Similarly the Eigen vector corresponding to $\lambda = -\sqrt{5}$, is $(\sqrt{5}-1, -1, 1)$.

Writing the three Eigen vectors as the three columns, we get the transformation (modal) matrix as $\boldsymbol{P} = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Hence the diagonal matrix is $\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$.

Q54. Find the matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to the diagonal form. Hence calculate A^4.

⇒ The Eigen values of $\boldsymbol{A}$ are $-2,3,6$ and Eigen vectors are $(-1,0,1), (1,-1,1), (1,2,1)$. Writing these Eigen vectors as three columns, the requied transformation matrix (modal matrix) is, $\boldsymbol{P} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

To find $\boldsymbol{P}^{-1}$, $\quad |\boldsymbol{P}| = \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (say)

$A_1 = -3, B_1 = 2, C_1 = 1, A_2 = 0, B_2 = -2, C_2 = 2, A_3 = 3, B_3 = 2, C_3 = 1$

Also $|\boldsymbol{P}| = a_1A_1 + b_1B_1 + c_1C_1 = 6 \qquad \therefore \boldsymbol{P}^{-1} = \frac{1}{|P|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Thus $\quad \boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad \therefore \boldsymbol{D}^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}$

Hence $\quad \boldsymbol{A}^4 = \boldsymbol{P}\boldsymbol{D}^4\boldsymbol{P}^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -8 & 0 & 8 \\ 27 & -27 & 27 \\ 216 & 512 & 216 \end{bmatrix} = \begin{bmatrix} 251 & 485 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{bmatrix}$

Q41. Find e^A & 4^A if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$.

⇒ The characteristic equation of $\boldsymbol{A}$ is $\begin{vmatrix} \frac{3}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2}-\lambda \end{vmatrix} = 0$ i.e., $\left(\frac{3}{2}-\lambda\right)^2 - \frac{1}{4} = 0 \qquad \therefore \lambda^2 - 3\lambda + 2 = 0$ Whence $\lambda = 1,2$.

When $\lambda = 1$, $[\boldsymbol{A} - \lambda\boldsymbol{I}]X = 0$, gives $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad or, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $[By\ 2R_1, 2R_2] \quad or, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $[By\ R_1 - R_2]$

$\therefore x_1 + x_2 = 0$. If $x_2 = -1, x_1 = 1$, i.e., the Eigen vector is $[1,-1]'$.

When $\lambda = 2$, $[\boldsymbol{A} - \lambda\boldsymbol{I}]X = 0$, gives $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad or, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $[By\ 2R_1, 2R_2] \quad or, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $[By\ R_1 - R_2]$

$\therefore -x_1 + x_2 = 0 \Rightarrow x_1 = x_2$. If $x_2 = 1, x_1 = 1$, i.e., the Eigen vector is $[1,1]'$.

Now $\boldsymbol{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ & $\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad\therefore \boldsymbol{P}^{-1} = \frac{adj\,\boldsymbol{P}}{|\boldsymbol{P}|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

If $f(\boldsymbol{A}) = e^{\boldsymbol{A}}$, $f(\boldsymbol{D}) = e^{\boldsymbol{D}} = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$ $\quad\therefore e^{\boldsymbol{A}} = Pf(\boldsymbol{D})\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{bmatrix}$

Replacing e by 4, we get $4^{\boldsymbol{A}} = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

Q55. Consider $\boldsymbol{\sigma_x} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{0} \end{pmatrix}$, $\boldsymbol{\sigma_y} = \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \boldsymbol{0} \end{pmatrix}$ <u>. which of the matrices are/ is Hermitian? Which of them are/is unitrary? Find</u> $\boldsymbol{\sigma_x\sigma_y}$ and $\boldsymbol{\sigma_y\sigma_x}$.

$\Rightarrow {\sigma_x}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$ again ${\sigma_y}^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$ Hence both matrices are Hermitian .

${\sigma_x}^\dagger\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Similarly ${\sigma_y}^\dagger \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Again $\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\sigma_y\sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. Note that hence $\sigma_x\sigma_y \neq \sigma_y\sigma_x$.

Q56. Consider $\boldsymbol{\sigma_1} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{0} \end{pmatrix}$, $\boldsymbol{\sigma_2} = \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \boldsymbol{0} \end{pmatrix}$ and $\boldsymbol{\sigma_3} = \begin{pmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{1} \end{pmatrix}$ and $[\boldsymbol{\sigma_i}, \boldsymbol{\sigma_j}] = \boldsymbol{\sigma_i\sigma_j} - \boldsymbol{\sigma_j\sigma_i}$

then show that $[\boldsymbol{\sigma_1}, [\boldsymbol{\sigma_2}, \boldsymbol{\sigma_3}]] + [\boldsymbol{\sigma_2}, [\boldsymbol{\sigma_3}, \boldsymbol{\sigma_1}]] + [\boldsymbol{\sigma_3}, [\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}]] = \boldsymbol{0}$

$\Rightarrow [\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\,\sigma_3$

Similarly $[\sigma_3, \sigma_1] = 2i\,\sigma_2$ and $[\sigma_2, \sigma_3] = 2i\,\sigma_1$

Then $[\sigma_1, [\sigma_2, \sigma_3]] + [\sigma_2, [\sigma_3, \sigma_1]] + [\sigma_3, [\sigma_1, \sigma_2]] = [\sigma_1, 2i\,\sigma_1] + [\sigma_2, 2i\,\sigma_2] + [\sigma_3, 2i\,\sigma_3] = 2i\,[\sigma_1, \sigma_1] + 2i[\sigma_2, \sigma_2] + 2i[\sigma_3, \sigma_3] = 0$