Electromagnetic theory

1. Generalization of Ampere's law

Displacement current, Maxwell's field of equation, wave equation for the electromagnetic fields and its solution plane wave and spherical wave solutions, transverse nature of field, relation between $\vec{E}$ & $\vec{B}$, energy density of field, Poynting vector and Poynting theorem, boundary conditions.

2. EM wave in an isotropic di-electric

Wave equation reflection and refraction at plane boundary reflection and transmission coefficient, Fresnel's formula, change of phase on reflection, polarisation on reflection and Brewster's law, total internal reflection.

3. EM wave in conducting medium

Wave equation in conducting medium, reflection and transmission at metallic surface-skin effect and skin depth, propagation of EM waves between parallel and conducting plates-wave guides (rectangular only).

4. Dispersion

Equation of motion of an electron in a radiation field: Lorentz theory of dispersion-normal and anomalous; Sellmeier's and Cauchy's formula, absorptive and dispersive mode, half power frequency, bandwidth

5. Scattering

Scattering of radiation by a bound charge, Raleigh scattering (qualitative ideas), blue of the sky, absorption

1. Generalization of Ampere's law

Displacement current, Maxwell's field of equation, wave equation for the electromagnetic fields and its solution plane wave and spherical wave solutions, transverse nature of field, relation between$\vec{E}$ & $\vec{B}$, energy density of field, Poynting vector and Poynting theorem, boundary conditions.

Q. Show that Ampere circuital law is valid in case where charge density is static that is steady state condition of charge flow so that ρ=constant.

Ampere's circuital law in most general form is given by $\oint \vec{B}.\vec{dl} = \mu_0 \iint \vec{J}.\vec{ds}$……(i)

where symbols have their usual meaning

Now by Stoke's law$\oint \vec{B}.\vec{dl} = \iint (\vec{\nabla} \times \vec{B}).\vec{ds}$ …….(ii)

then from equation (i) and (ii) $\iint (\vec{\nabla} \times \vec{B}).\vec{ds} = \mu_0 \iint \vec{J}.\vec{ds}$………(iii)

As equation (iii) is valid for any arbitrary surface $\vec{ds}$ then $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$…………..(iv)

Now $\vec{\nabla}.\vec{\nabla} \times \vec{B} = \vec{\nabla}.\mu_0 \vec{J} = \mu_0 \vec{\nabla}.\vec{J}$ As $\vec{\nabla}.\vec{\nabla} \times \vec{B} = 0$ always then we have $\vec{\nabla}.\vec{J} = 0$

Then from continuity equation $\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$ we get $\frac{\partial \rho}{\partial t} = 0$

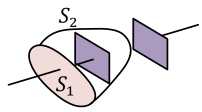
hence ampere circuital law is valid only when Charge density is static.

From this situation we conclude that ampere circuital law in equation (i) or this law in differential form in equation (iv) for if not complete and not valid for the case of time dependent field.

Q. Show that Ampere circuital law must be modified with the inclusion of the effect of Capacitor for a circuit consisting of a parallel plate capacitor

Suppose that the capacitor is charged by a current I. Consider a closed path C bounding the surface S_1& S_2. The surface S_1 intersecting a lead carrying the current I and the surface S_2 passing through the region between the capacitor plates. Ampere's law for closed path C and for the surface S_1 yields $\oint \vec{H}.\vec{dl} = \iint \vec{J}.\vec{ds} = I$……(I)

The same law for the close path C and for surface S_2 yields $\oint \vec{H}.\vec{dl} = \iint \vec{J}.\vec{ds} = 0$……(II)



Equation (I) and (II) are contradictory equation (I) is correct as it does not involve the capacitor.

On the other hand ampere's law must be modified with the inclusion of effect of capacitor.

Q. How Maxwell cure the defect in ampere's law

Ampere's law in differential form $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ does not hold good for in case of time varying current, where $\frac{\partial \rho}{\partial t} \neq 0$

To cure the defect Maxwell proposed that something is to be added to right hand side of the equation$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

so let $\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_D)$

Now to investigate the significance of $\vec{J}_D$ we have $\vec{\nabla}.\vec{\nabla} \times \vec{B} = \vec{\nabla}.\mu_0 (\vec{J} + \vec{J}_D) = \mu_0 \vec{\nabla}.(\vec{J} + \vec{J}_D)$

As $\vec{\nabla}.\vec{\nabla} \times \vec{B} = 0$ always then we have $\vec{\nabla}.(\vec{J} + \vec{J}_D) = 0$

or, $\vec{\nabla}.\vec{J} = -\vec{\nabla}.\vec{J}_D$ or, $-\frac{\partial \rho}{\partial t} = -\vec{\nabla}.\vec{J}_D$ or, $-\frac{\partial (\vec{\nabla}.\vec{D})}{\partial t} = -\vec{\nabla}.\vec{J}_D$ [as ρ=$\vec{\nabla}.\vec{D}$]

or, $\vec{\nabla}.\left(\vec{J}_D - \frac{\partial}{\partial t}\vec{D}\right)$=0 hence $\vec{J}_D = \frac{\partial}{\partial t}\vec{D}$ this term is called Displacement current density.

Then modified form of ampere's circuital law is$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial}{\partial t}\vec{D}\right)$

Q. Define displacement current

First of all displacement current density is defined as the rate of change of electric displacement vector with time, i.e., $\vec{J}_D = \frac{\partial}{\partial t}\vec{D}$. Displacement current may be defined as the current arising due to time varying electric field between the plate of a capacitor. The total displacement current is $I_D = \iint \vec{J}_D.\overrightarrow{ds}$ where $\overrightarrow{ds}$ is any elementary surface

Q. Is displacement current related to the motion of charge flow

No displacement current is a current only in the sense that it produced in magnetic field it has none of the other properties of current as it is not related to the motion of charged. So displacement current density $\vec{J}_D = \frac{\partial}{\partial t}\vec{D}$ has certain values when even $\vec{J} = 0$

Q. Distinction between conduction current and displacement current

Conduction current	**Displacement current**
1. it is due to actual flow of charges in a conductor 2. Conduction current density is given by $\vec{J} = \sigma\vec{E}$ where $\vec{E}$ =electric field σ= conductivity 3. conduction current obey Ohm's law 4. In good conductor $\vec{J} \gg \vec{J}_D$. i.e., $\vec{J}_D \to 0$ 5. $\vec{J}$ Represents the power of conduction of a conductor .	1. It arises due to time varying electric field in a dielectric 2. Displacement current is given by $\vec{J}_D = \frac{\partial}{\partial t}\vec{D} = \epsilon\frac{\partial}{\partial t}\vec{E}$ where ϵ = permittivity of the medium 3. It does not obey Ohm's law 4. In dielectric $\vec{J}_D \gg \vec{J}$ 5. $\vec{J}_D$ Represent the permittivity power of a dielectric

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1. **Q. What are the consequence of the introduction of $\vec{J}_D$**

The symmetry character of electric field and magnetic field is more prominent after the introduction of $\vec{J}_D$. Now then a changing electric field is now seen to produce magnetic field just as a change in magnetic field gives rise to electric field.

With the introduction of $\vec{J}_D$ both steady and non-steady current circuit can be analysed that is the situation where

$\vec{\nabla}.\vec{J} \neq 0$ All the variation of AC circuits with capacitor can be understood on the basis of $\vec{J}_D$

Q. Prove that displacement current in a dielectric a parallel plate capacitor is equal to the conduction current in the connecting leads (numerically)

$$\vec{J}_D = \frac{\partial}{\partial t}\vec{D} = \epsilon\frac{\partial}{\partial t}\vec{E} = \epsilon\frac{\partial}{\partial t}\left(\frac{\sigma}{\epsilon}\hat{n}\right) = \frac{\partial\sigma}{\partial t}\hat{n}$$

Then $I_D = \iint \vec{J}_D.\overrightarrow{ds} = \iint \frac{\partial\sigma}{\partial t}ds = \frac{\partial\sigma}{\partial t}S = \frac{\partial(\sigma S)}{\partial t} = \frac{\partial q}{\partial t} = I$ (proved)

Q. Show that $I_D = \epsilon\frac{\partial\Phi_e}{\partial t}$ where Φ_e=electric flux over a surface S

$I_D = \iint \vec{J}_D.\overrightarrow{ds} = \iint \frac{\partial}{\partial t}\vec{D}.\overrightarrow{ds} = \iint \epsilon\frac{\partial}{\partial t}\vec{E}.\overrightarrow{ds} = \epsilon\iint\frac{\partial}{\partial t}\vec{E}.\overrightarrow{ds} = \epsilon\frac{\partial}{\partial t}\iint\vec{E}.\overrightarrow{ds} = \epsilon\frac{\partial\Phi_e}{\partial t}$ (proved)

Q. prove that the displacement current density leads the conduction current density by $\frac{\pi}{2}$

Let an alternating field $\vec{E} = \overrightarrow{E_0}\cos\omega t$ then $\vec{J} = \sigma\vec{E} = \sigma\overrightarrow{E_0}\cos\omega t$

And $\vec{J}_D = \frac{\partial}{\partial t}\vec{D} = \epsilon\frac{\partial}{\partial t}\vec{E} = -\epsilon\omega\,\overrightarrow{E_0}\sin\omega t = \epsilon\omega\overrightarrow{E_0}\cos(\omega t + \frac{\pi}{2})$ (proved)

Maxwell's electromagnetic equation.

Integral form

1. Gauss's law in electrostatic: $\oiint \vec{E}.\overrightarrow{ds} = \frac{q}{\epsilon_0}$ for free space.
2. Gauss's law in magneto static $\oiint \vec{B}.\overrightarrow{ds} = 0$
3. Faraday's law of electromagnetic induction: $\iint_S(\vec{\nabla}\times\vec{E}).\overrightarrow{dS} = -\iint_S\frac{\partial\vec{B}}{\partial t}.\overrightarrow{dS}$
4. Modified form of Ampere's circuital law. $\oint\vec{B}.\overrightarrow{dl} = \mu_0 I = \mu_0\iint(\vec{J}+\vec{J}_D).\overrightarrow{ds} = \mu_0\iint\left(\vec{J}+\frac{\partial}{\partial t}\vec{D}\right).\overrightarrow{ds}$

Differential form:

1. Gauss's law in electrostatic: (i) $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ for free space, (ii) $\vec{\nabla}.\vec{D} = \rho$ for any dielectric medium.
2. Gauss's law in magneto static: $\vec{\nabla}.\vec{B} = 0$
3. Faraday's law of electromagnetic induction: $\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial\vec{B}}{\partial t}$
4. Modified form of Ampere's circuital law. $\left(\vec{\nabla}\times\vec{B}\right) = \mu_0\left(\vec{J}+\frac{\partial}{\partial t}\vec{D}\right)$

Physical significance of Maxwell's equation

1. $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ The equation represents that the electric lines of force are not closed. The line of force originated from the positive charges source and terminates in the negative charges sink. Note that this equation is time independent. **Integral form indicates total electric flux is equal to charge enclosed by the surface.**

2. $\vec{\nabla}.\vec{B} = 0$This equation represent that the magnetic lines of force are closed. There is no source or sink in magnetic flux in the volume. That means magnetic monopole does not exist. Note that this equation is also time independent. **Integral form indicates magnetic flux is always zero.**
3. $\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial\vec{B}}{\partial t}$ This equation is time dependent hence change in magnetic field can be produced a change in electric field. **Integral form indicates induce EMF in a closed circuit is equal to rate of change of magnetic flux**

4. $\left(\vec{\nabla}\times\vec{B}\right) = \mu_0\left(\vec{J}+\frac{\partial}{\partial t}\vec{D}\right)$This equation is time dependent tells us change in electric field produced a change in magnetic field **this equation gives local conservation of charge.**

Properties of Maxwell equation

1. Maxwell's equations are linear. That means principle of superposition is valid here.
2. Equations include the equation of continuity hence Maxwell equation is consistent of the local charge conservation law
3. Maxwell's equations are relativistic invariant
4. Maxwell's equation is not symmetric with respect to electric and magnetic field general but taken up symmetric form in free space where $\vec{J} = 0,\ \rho = 0$
5. Maxwell equation predict the existence of electromagnetic wave.

Maxwell's equation in different medium:

a. Free space: Here $\rho = 0,\ \sigma = 0,\ \vec{D} = \epsilon_0 \vec{E}, \vec{B} = \mu_0 \vec{H}$
1. $\vec{\nabla}.\vec{E} = 0$ or, $\vec{\nabla}.\vec{D} = 0$
2. $\vec{\nabla}.\vec{B} = 0$ or, , $\vec{\nabla}.\vec{H} = 0$
3. $(\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial \vec{H}}{\partial t}$
4. $(\vec{\nabla} \times \vec{H}) = \epsilon_0 \frac{\partial}{\partial t} \vec{E}$ [As $\vec{J} = \sigma \vec{E} = 0\ as\ \sigma = 0$]

b. Conducting medium: Here $\rho = 0$, As there is equal number of positive and negative charge. $\vec{J}_D = 0$
1. $\vec{\nabla}.\vec{E} = 0$ or, $\vec{\nabla}.\vec{D} = 0$
2. $\vec{\nabla}.\vec{B} = 0$ or, , $\vec{\nabla}.\vec{H} = 0$
3. $(\vec{\nabla} \times \vec{E}) = \frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$
4. $(\vec{\nabla} \times \vec{H}) = \vec{J} = \sigma \vec{E}$ [As $\vec{J}_D = 0$]

c. Dielectric medium : Here $\rho = 0,\ \sigma = 0, \vec{J} = \sigma \vec{E} = 0\ \ \vec{D} = \epsilon \vec{E}, \vec{B} = \mu \vec{H}$
1. $\vec{\nabla}.\vec{E} = 0$ or, $\vec{\nabla}.\vec{D} = 0$
2. $\vec{\nabla}.\vec{B} = 0$ or, , $\vec{\nabla}.\vec{H} = 0$
3. $(\vec{\nabla} \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$
4. $(\vec{\nabla} \times \vec{H}) = \frac{\partial \vec{B}}{\partial t}$ [As $\vec{J} = 0$]

Note that Maxwell's equation for static electric and magnetic field is $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ or, $\vec{\nabla}.\vec{D} = \rho$

$\vec{\nabla}.\vec{B} = 0$ or, , $\vec{\nabla}.\vec{H} = 0$

$(\vec{\nabla} \times \vec{E}) = 0$

$(\vec{\nabla} \times \vec{H}) = \vec{J} = \sigma \vec{E}$

Conservation of electromagnetic energy-Poynting's theorem

Poynting's theorem is a statement of conservation of energy applied to electromagnetic fields. To establish the theorem let us start from Maxwell's equations $(\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial \vec{H}}{\partial t}$. Now

$$\vec{\nabla}.(\vec{E} \times \vec{H}) = \vec{H}.(\vec{\nabla} \times \vec{E}) - \vec{E}.(\vec{\nabla} \times \vec{H}) = \vec{H}.\left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{E}.\left(\vec{J} + \frac{\partial}{\partial t}\vec{D}\right) = -\vec{H}.\left(\frac{\partial \vec{B}}{\partial t}\right) - \vec{E}.\vec{J} - \vec{E}.\frac{\partial}{\partial t}\vec{D}$$

For any linear medium $\vec{D} = \epsilon \vec{E}$ & $\vec{B} = \mu \vec{H}$. So we can rewrite the above equation as

$$\vec{\nabla}.(\vec{E} \times \vec{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}.\vec{D} + \frac{1}{2}\vec{B}.\vec{H}\right) - \vec{E}.\vec{J}$$

Integrating the equation over a fixed volume V bounded by a closed surface S and applying Gausses Divergence theorem $\oint_S (\vec{E} \times \vec{H}).\vec{dS} = -\frac{d}{dt}\iiint_V \left(\frac{1}{2}\vec{E}.\vec{D} + \frac{1}{2}\vec{B}.\vec{H}\right) dV - \iiint_V (\vec{E}.\vec{J}) dV$

$$\text{or, } -\frac{d}{dt}\iiint_V\left(\frac{1}{2}\vec{E}.\vec{D}+\frac{1}{2}\vec{B}.\vec{H}\right)dV=\iiint_V(\vec{E}.\vec{J})dV+\oint_S(\vec{E}\times\vec{H}).\overrightarrow{dS} \quad \text{..............................(i)}$$

To understand the physical significance of equation(i) let us interpret its term in it.

Suppose we have some charge and current configuration which at time t produce the $\vec{E}$ & $\vec{B}$ fields.

The rate of one ton by the electromagnetic forces on an element of charge $dq = \rho dV$ is given by

$$\frac{\mathrm{dW}}{\mathrm{dt}} = dq(\vec{E}+\vec{v}\times\vec{B}).\vec{v} = dq(\vec{E}).\vec{v} = (\vec{E}).\vec{v}\,\rho dV = (\vec{E}.\vec{J})dV$$

[where $\vec{v}$ is the velocity of charge element and $\vec{J} = \vec{v}\,\rho$]

Thus the term $\iiint_V(\boldsymbol{\vec{E}.\vec{J}})\boldsymbol{dV}$ represents the **rate of doing work on the charges in volume V by the electromagnetic field.**

Again we know $\frac{1}{2}\vec{E}.\vec{D}$ if the electrostatic energy density $\frac{1}{2}\vec{B}.\vec{H}$ is the magneto-static energy density

Thus the term $\left(\frac{1}{2}\vec{E}.\vec{D}+\frac{1}{2}\vec{B}.\vec{H}\right)$ Maybe interpreted as electromagnetic energy density.

Thus the term $-\frac{d}{dt}\iiint_V\left(\frac{1}{2}\vec{E}.\vec{D}+\frac{1}{2}\vec{B}.\vec{H}\right)dV$ represents the rate at which the total electromagnetic energy in volume V is decreasing. Hence physical meaning of the term $\oint_S(\vec{E}\times\vec{H}).\overrightarrow{dS}$ now follows from the principle of conservation of energy.

The rate of decrease of electromagnetic field energy within a certain volume is to equal to the rate of work done by the field on the charges insight the given volume plus the rate of outflow of electromagnetic energy through the surface bounding the volume. This statement is known as Poynting's theorem. So the term $\oint_S(\vec{E}\times\vec{H}).\overrightarrow{dS}$

Represents the rate of flow of electromagnetic energy out towards through the surface S. The vector $\vec{s} = \vec{E}\times\vec{H}$ represents the amount of electromagnetic energy flowing out normally through unit area but unit time. This vector

$\vec{s} = \vec{E}\times\vec{H}$ is called Poynting's vector.

<u>Differential form of Poynting's theorem:</u>

The work done by the electromagnetic field on the charges increased their mechanical energy. If we denote mechanical energy density by u_M then we can write $\frac{\partial u_M}{\partial \mathrm{t}}$ = rate of work done on charges per unit volume $= \vec{E}.\vec{J}$

now denoting the electromagnetic energy density $\left(\frac{1}{2}\vec{E}.\vec{D}+\frac{1}{2}\vec{B}.\vec{H}\right)$ By u_{em} we can write equation (i)

$$\frac{\partial}{\partial \mathrm{t}}(u_M + u_{em}) + \vec{\nabla}.\vec{s} = 0$$

this is the differential form of Poynting's theorem. This has the same form as the equation of continuity expressing the conservation of charge, with the total energy density taking the place of charge density ρ and $\vec{s}$ taking the place of $\vec{J}$. Therefore, from the analogy with $\vec{J}$ the **Poynting's vector can be interpreted as the energy flowing through unit area per unit time.**

Wave equation in free space:

in free space, where there is no charge (ρ=0) or no current ($\vec{J}$=0) Maxwell's equation take the following form

$\vec{\nabla}.\vec{E} = 0$

$\vec{\nabla}.\vec{H} = 0$

$(\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$(\vec{\nabla} \times \vec{H}) = \epsilon_0 \frac{\partial}{\partial t}\vec{E}$ [As $\vec{J} = \sigma\vec{E} = 0 \; as \; \sigma = 0$]

Taking curl of equation $(\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ we get $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu_0 \frac{\partial \vec{H}}{\partial t}\right)$

Or, $\vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E} = \mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$

Or, $-\nabla^2\vec{E} = \mu_0 \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\partial}{\partial t}\vec{E}\right)$

Or, $\nabla^2\vec{E} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} = \vec{0}$……………………..(ii)

Similarly Taking curl of equation$(\vec{\nabla} \times \vec{H}) = \epsilon_0 \frac{\partial}{\partial t}\vec{E}$ we get $\nabla^2\vec{H} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{H} = \vec{0}$…………..(iii)

thus both $\vec{E}$ & $\vec{H}$ satisfied the well-known wave equation $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0$. thus both $\vec{E}$ & $\vec{H}$ propagate in free space in the form of the wave. And comparing with the wave equation we find the velocity of propagation $c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$

This indicates that the light is an electromagnetic wave.

Plane wave solutions:

the simplest solution of equation(ii) &(iii) are plane wave solutions

$\vec{E}(\vec{r},t) = \overrightarrow{E_0}e^{-i(\vec{k}.\vec{r}-\omega t)}$, $\vec{H}(\vec{r},t) = \overrightarrow{H_0}e^{-i(\vec{k}.\vec{r}-\omega t)}$

Where wave vector $\vec{k} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\upsilon}{c}\hat{n} = \frac{\omega}{c}\hat{n}$ here $\hat{n}$ is the unit vector along the direction of propagation and $\vec{E}_0, \vec{H}_0$ are complex amplitudes.

Spherical wave solutions:

because of the vector nature of $\boldsymbol{\vec{E}}$ & $\boldsymbol{\vec{H}}$ **it is very difficult to find the spherical wave solution of wave equation. If we assume that the field parameter depends only the radial coordinate r of spherical coordinate systems then the wave equation reduces to** $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$

Putting $\psi = \frac{\psi'}{r}$ we get $\frac{\partial^2\psi'}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2\psi'}{\partial t^2} = 0$

the general solution of the above question is $\psi' = f_1(ct-r) + f_2(ct+r)$

or, $\psi = \frac{f_1(ct-r)}{r} + \frac{f_2(ct+r)}{r}$

The first-term in the right hand side represents a spherical wave diverging from origin of the coordinate system with the velocity c and second term represents a spherical wave converging to the origin.

A diverging spherical harmonic wave can be represented as $\psi(r,t) = A\frac{1}{r}\cos(kr-\omega t) + B\frac{1}{r}\sin(kr-\omega t)$

The relative directions of $\vec{E}$ & $\vec{H}$ and $\vec{k}$

Substituting the solutions $\vec{E}(\vec{r},t) = \overrightarrow{E_0} e^{-i(\vec{k}.\vec{r}-\omega t)}$, $\vec{H}(\vec{r},t) = \overrightarrow{H_0} e^{-i(\vec{k}.\vec{r}-\omega t)}$ In equations $\vec{\nabla}.\vec{E} = 0$ & $\vec{\nabla}.\vec{H} = 0$

we get $\vec{k}.\vec{E} = 0$ & $\vec{k}.\vec{H} = 0$ thus both $\vec{E}$ & $\vec{H}$ are perpendicular to $\vec{k}$. This implies that electromagnetic waves are transvers in nature.

Again From $(\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ we get $\boldsymbol{j\vec{k} \times \vec{E} = -\mu_0(-j\omega\vec{H})}$ **or,** $\boldsymbol{\vec{k} \times \vec{E} = \mu_0 \omega \vec{H}}$**……….(iv)**

And from $(\vec{\nabla} \times \vec{H}) = \epsilon_0 \frac{\partial}{\partial t}\vec{E}$ we get $\boldsymbol{j\vec{k} \times \vec{H} = -\epsilon_0(-j\omega\vec{E})}$ **or,** $\boldsymbol{\vec{k} \times \vec{H} = \epsilon_0 \omega \vec{E}}$**……………..(v)**

from equation (iv) it is clear that $\vec{H}$ if perpendicular to both $\vec{k}$ & $\vec{E}$. from equation (v) it is clear that $\vec{E}$ if perpendicular to both $\vec{k}$ & $\vec{H}$. Thus the field vectors $\vec{E}$ & $\vec{H}$ and mutually perpendicular and also they are perpendicular to the direction of propagation $\vec{k}$. The vectors $\vec{E}$ & $\vec{H}$ & $\vec{k}$ form if set of orthogonal vectors, which constitutes a right handed system in that order.

Relative phase of $\vec{E}$ & $\vec{H}$

From $\nabla^2\vec{E} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} = \vec{0}$ or from $\nabla^2\vec{H} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{H} = \vec{0}$ we get $\boldsymbol{k^2 = \mu_0 \epsilon_0 \omega^2}$

thus, in free-space k is real quantity and hence implies that $\vec{E}$ & $\vec{H}$ **are in phase**

Wave impedance:

the ratio of magnitude of $\vec{E}$ & $\vec{H}$ is $Z_0 = \frac{|\vec{E}|}{|\vec{H}|} = \frac{\mu_0\omega}{k} = \sqrt{\frac{\mu_0}{\epsilon_0}}$

As Z_0 has the dimensions of impedance and is known as the wave impedance in free $\sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}}} = 376.6\Omega$

Poynting's vector

$$\vec{s} = \vec{E} \times \vec{H} = \vec{E} \times \left(\frac{1}{\mu_0\omega}\vec{k} \times \vec{E}\right) = \frac{1}{\mu_0\omega}\left[\vec{k}(\vec{E}.\vec{E}) - \vec{E}(\vec{E}.\vec{k})\right] = \frac{1}{\mu_0\omega}E^2\vec{k} = \frac{E^2}{Z_0 k}\vec{k} = \frac{E^2}{Z_0}\hat{n}$$

Thus the energy flow in the direction of where propagation. Since $\vec{E}$ is normal to $\vec{k}$, then we can write from equation(iv) $kE = \mu_0\omega H$ or, $E = \frac{\mu_0\omega}{k}H$ or, $E = \sqrt{\frac{\mu_0}{\epsilon_0}}H$ or, $\sqrt{\epsilon_0}\, E = \sqrt{\mu_0}H$ or, $\frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\mu_0 H^2$

This shows that in case of electromagnetic wave in free space energy is equally shared between electric and magnetic fields. Hence total electromagnetic energy density $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 = \epsilon_0 E^2$

or, $E^2 = \frac{u}{\epsilon_0}$ Then Poynting's vector can be written as $\vec{s} = \frac{\frac{u}{\epsilon_0}}{Z_0}\hat{n} = \frac{u}{\epsilon_0\sqrt{\frac{\mu_0}{\epsilon_0}}}\hat{n} = u\sqrt{\frac{1}{\mu_0\epsilon_0}}\hat{n} = uc\,\hat{n}$

Time average of Poynting's vector:

So far we have freely used the complex solution of the field vectors $\vec{E}$ & $\vec{H}$ **with the understanding that actual quantities are given by real parts of the complex solutions. As** $\vec{s}$ is non-linear in the fields it is essential to take a real parts of the fields before multiplying them.

Thus the real point in vector $\vec{s} = \mathrm{Re}\vec{E} \times \mathrm{Re}\vec{H}$ where $\mathrm{Re}\vec{E} = \frac{1}{2}(\vec{E} + \vec{E}^*)$ and $\mathrm{Re}\vec{H} = \frac{1}{2}(\vec{H} + \vec{H}^*)$

$$\vec{s} = \frac{1}{4}\left[\left(\vec{E} + \vec{E}^*\right) \times \left(\vec{H} + \vec{H}^*\right)\right] = \frac{1}{4}\left[\vec{E} \times \vec{H} + \vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} + \vec{E}^* \times \vec{H}^*\right]$$

$$\langle\vec{s}\rangle = \frac{1}{4}\left[\langle\vec{E} \times \vec{H}\rangle + \langle\vec{E} \times \vec{H}^*\rangle + \langle\vec{E}^* \times \vec{H}\rangle + \langle\vec{E}^* \times \vec{H}^*\rangle\right] = \frac{1}{4}\left[\langle\vec{E} \times \vec{H}^*\rangle + \langle\vec{E}^* \times \vec{H}\rangle\right]$$

$$\langle\vec{s}\rangle = \frac{1}{4}\left[\left(\vec{E} \times \vec{H}^*\right) + \left(\vec{E}^* \times \vec{H}\right)\right] = \frac{1}{2}Re\left(\vec{E} \times \vec{H}^*\right) = \frac{1}{2}Re\left(\vec{E}^* \times \vec{H}\right)$$

the magnitude of time average of pointing vector is called intensity of radiation(I). Thus the intensity

$$I = |\langle\vec{s}\rangle| = \frac{1}{2}E_0H_0 = \frac{E_0}{\sqrt{2}}\frac{H_0}{\sqrt{2}} = E_{r.m.s} \times H_{r.m.s}$$

Similarly, the time average of energy density

$$\langle u\rangle = \langle\epsilon_0\, E^2\rangle = \langle\epsilon_0\, \vec{E}.\vec{E}\rangle = \frac{1}{2}\epsilon_0\, Re\left(\vec{E}.\vec{E}^*\right) = \frac{1}{2}\epsilon_0\, {E_0}^2 = \epsilon_0\, {E^2}_{r.m.s}$$

Relation between intensity and time average of energy density

$$\frac{I}{\langle u\rangle} = \frac{1}{\epsilon_0}\frac{H_{r.m.s}}{E_{r.m.s}} = \sqrt{\frac{1}{\mu_0\epsilon_0}} = c \qquad \text{or, } I = c\,\langle u\rangle$$

Boundary conditions

when electromagnetic waves exist in more than one medium the field vectors must satisfy certain conditions at the interface of two media. These conditions are known as boundary condition. These conditions can be deduced from Maxwell's equation.

To find the boundary conditions for we note that Maxwell's diverges equations

$$\vec{\nabla}.\vec{D}(t) = \rho(t) \qquad \vec{\nabla}.\vec{B}(t) = 0$$

are exactly of the same form as the corresponding equations in electrostatic and magneto-static except that here$\vec{D}(t)$, $\vec{B}(t)$, $\rho(t)$ are time dependent.

To find the boundary condition on $\vec{D}$ we consider figure 1. Integrating the equation $\vec{\nabla}.\vec{D}(t) = \rho(t)$ over the volume of the Pill-box and using divergence theorem we get $\iint_S \vec{D}.\hat{n}dS = \sigma\Delta S$

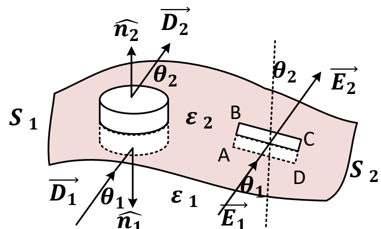


Fig 1

Now following the procedure as for electrostatic and magneto-static case

we get $\overrightarrow{D_2}.\hat{n} - \overrightarrow{D_1}.\hat{n} = \sigma$……….(I)

so the normal components of $\vec{D}$ are discontinuous across the interface by an amount equal to the free surface charge density.

Similarly again following procedure as before for static case we get $\overrightarrow{B_2}.\hat{n} = \overrightarrow{B_1}.\hat{n}$ ……………..(II)

Thus, the normal components of $\vec{B}$ are continuous across the interface.

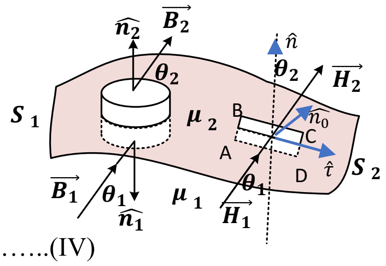
Maxwell's Curl equations $\vec{\nabla} \times \vec{H} = \left(\vec{J} + \frac{\partial}{\partial t}\vec{D}\right)$ and $\left(\vec{\nabla} \times \vec{E}\right) = -\frac{\partial\vec{B}}{\partial t}$ are different from the corresponding static equations. However, it turns out that the boundary conditions remain exactly the same as in static case with the time dependence retained in the fields and the current. To find the condition on the tangential components of $\vec{H}$ we integrate $\vec{\nabla} \times \vec{H} = \left(\vec{J} + \frac{\partial}{\partial t}\vec{D}\right)$ Over the surface of small rectangle shown in figure 2. Thus

$$\iint_{\Delta S}(\vec{\nabla} \times \vec{H}).\hat{n}dS = \iint_{\Delta S}\vec{J}.\hat{n}dS + \iint_{\Delta S}\frac{\partial}{\partial t}\vec{D}.\hat{n}dS$$

If $\frac{\partial}{\partial t}\vec{D}$ is bounded, then it's integral over ΔS vanishes in the limit $\Delta h \to 0$. So following the procedure as it was in the static case we get $\hat{n} \times \overrightarrow{H_2} - \hat{n} \times \overrightarrow{H_2} = \vec{K}$……….(III)

Show the tangential component of$\vec{H}$ are discontinuous across the interface by an amount equal to the free surface current density.

A similar integration of the equation $(\vec{\nabla} \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}$ over the surface of small rectangle of figure 2 gives



$\iint_{\Delta S} (\vec{\nabla} \times \vec{E}).\,\hat{n} dS = -\iint_{\Delta S} \frac{\partial}{\partial t} \vec{B}.\,\hat{n} dS$

$\oint_C \vec{E}.\,\vec{dl} = -\iint_{\Delta S} \frac{\partial}{\partial t} \vec{B}.\,\hat{n} dS$ [using Stokes' theorem]

If $\frac{\partial}{\partial t} \vec{B}$ is bounded, then it's integral over ΔS vanishes in the limit

$\Delta h \to 0$. So following the procedure as it was in the static case we get $\overrightarrow{E_2}.\,\hat{\tau} = \overrightarrow{E_1}.\,\hat{\tau}$………….(IV)

Thus the tangential component of $\vec{E}$ are continuous across the boundary.

Equation (I)-(IV) constitute the most general boundary conditions for the electromagnetic fields. In case of linear medium the boundary conditions can be expressed in terms of $\vec{E}$& $\vec{B}$ fields

$$\epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \sigma$$

$$B_{2n} = B_{1n}$$

$$\hat{n} \times \frac{\overrightarrow{B_2}}{\mu_2} - \hat{n} \times \frac{\overrightarrow{B_1}}{\mu_1} = \vec{K}$$

$$E_{2t} = E_{1t}$$

2. EM wave in an isotropic di-electric

Wave equation reflection and refraction at plane boundary reflection and transmission coefficient, Fresnel's formula, change of phase on reflection, polarisation on reflection and Brewster's law, total internal reflection.

Plane electromagnetic wave in an isotropic dielectric medium.

In isotropic die electric medium , where there is no charge (ρ=0) or no current ($\vec{J}$=0) Maxwell's equation take the following form

$\vec{\nabla}.\vec{E} = 0$

$\vec{\nabla}.\vec{H} = 0$

$(\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial \vec{H}}{\partial t}$

$(\vec{\nabla} \times \vec{H}) = \epsilon \frac{\partial}{\partial t} \vec{E}$ [As $\vec{J} = \sigma \vec{E} = 0$ as $\sigma = 0$]

Taking curl of equation $(\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial \vec{H}}{\partial t}$ we get $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu \frac{\partial \vec{H}}{\partial t}\right)$

$$\text{Or, } \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$\text{Or, } -\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}\left(\epsilon \frac{\partial}{\partial t} \vec{E}\right)$$

$$\text{Or, } \nabla^2 \vec{E} - \mu \epsilon \frac{\partial^2}{\partial t^2} \vec{E} = \vec{0} \text{……………………..(ii)}$$

Similarly Taking curl of equation$(\vec{\nabla} \times \vec{H}) = \epsilon \frac{\partial}{\partial t} \vec{E}$ we get $\nabla^2 \vec{H} - \mu \epsilon \frac{\partial^2}{\partial t^2} \vec{H} = \vec{0}$…………..(iii) thus both $\vec{E}$ & $\vec{H}$ satisfied the well-known wave equation $\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi = 0$. thus both $\vec{E}$ & $\vec{H}$ propagate in an isotropic dielectric medium in the form of the wave. And comparing with the wave equation we find the velocity of propagation $v = \sqrt{\frac{1}{\mu\epsilon}}$ This indicates that the light in an isotropic dielectric medium is an electromagnetic wave.

Now $v = \sqrt{\frac{1}{\mu\epsilon}} = \sqrt{\frac{1}{\mu_0\epsilon_0\mu_r\epsilon_r}} = \frac{c}{\sqrt{\mu_r\epsilon_r}}$ For nonmagnetic dielectric medium $\mu_r = 1$ then $v = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{K}}$

Where K = relative permittivity or dielectric constant.

Plane wave solutions:

The simplest solution of equation(ii) &(iii) are plane wave solutions

$\vec{E}(\vec{r},t) = \overrightarrow{E_0}e^{-i(\vec{k}.\vec{r}-\omega t)}$, $\vec{H}(\vec{r},t) = \overrightarrow{H_0}e^{-i(\vec{k}.\vec{r}-\omega t)}$

Where wave vector $\vec{k} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi v}{c}\hat{n} = \frac{\omega}{c}\hat{n}$ here $\hat{n}$ is the unit vector along the direction of propagation and $\vec{E}_0, \vec{H}_0$ are complex amplitudes.

The relative directions of $\vec{E}$ & $\vec{H}$ and $\vec{k}$

Substituting the solutions $\vec{E}(\vec{r},t) = \overrightarrow{E_0}e^{-i(\vec{k}.\vec{r}-\omega t)}$, $\vec{H}(\vec{r},t) = \overrightarrow{H_0}e^{-i(\vec{k}.\vec{r}-\omega t)}$ in equations $\vec{\nabla}.\vec{E} = 0$ &$\vec{\nabla}.\vec{H} = 0$

we get $\vec{k}.\vec{E} = 0$ & $\vec{k}.\vec{H} = 0$ thus both $\vec{E}$ & $\vec{H}$ are perpendicular to $\vec{k}$. This implies that electromagnetic waves are transvers in nature.

Again From $(\vec{\nabla}\times\vec{E}) = -\mu\frac{\partial\vec{H}}{\partial t}$ we get $\boldsymbol{j\vec{k}\times\vec{E}} = -\mu(\boldsymbol{-j\omega\vec{H}})$ **or,** $\boldsymbol{\vec{k}\times\vec{E}} = \mu\boldsymbol{\omega\vec{H}}$**……….(iv)**

And from $(\vec{\nabla}\times\vec{H}) = \epsilon\frac{\partial}{\partial t}\vec{E}$ we get $\boldsymbol{j\vec{k}\times\vec{H}} = -\epsilon\,(\boldsymbol{-j\omega\vec{E}})$ **or,** $\boldsymbol{\vec{k}\times\vec{H}} = \epsilon\,\boldsymbol{\omega\vec{E}}$**……………..(v)**

from equation (iv) it is clear that $\vec{H}$ if perpendicular to both $\vec{k}$ & $\vec{E}$. from equation (v) it is clear that $\vec{E}$ if perpendicular to both $\vec{k}$ & $\vec{H}$. Thus the field vectors $\vec{E}$ & $\vec{H}$ and mutually perpendicular and also they are perpendicular to the direction of propagation $\vec{k}$. The vectors $\vec{E}$ & $\vec{H}$ & $\vec{k}$ form if set of orthogonal vectors, which constitutes a right handed system in that order.

Relative phase of $\vec{E}$ & $\vec{H}$

From $\nabla^2\vec{E} - \mu\,\epsilon\frac{\partial^2}{\partial t^2}\vec{E} = \vec{0}$ or from $\nabla^2\vec{H} - \mu\,\epsilon\frac{\partial^2}{\partial t^2}\vec{H} = \vec{0}$ we get $\boldsymbol{k^2} = \mu\,\epsilon\,\boldsymbol{\omega^2}$

thus, in an isotropic dielectric medium k is **real quantity** and hence implies that $\vec{E}$ & $\vec{H}$ **are in phase**

Wave impedance:

the ratio of magnitude of $\vec{E}$ & $\vec{H}$ is $Z = \frac{|\vec{E}|}{|\vec{H}|} = \frac{\mu\omega}{k} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}} = Z_0\sqrt{\frac{\mu_r}{\epsilon_r}}$

For nonmagnetic dielectric medium $\mu_r = 1$ then $Z = \frac{Z_0}{\sqrt{\epsilon_r}} = \frac{Z_0}{\sqrt{K}}$

Poynting's vector

$$\vec{s} = \vec{E}\times\vec{H} = \vec{E}\times\left(\frac{1}{\mu\omega}\vec{k}\times\vec{E}\right) = \frac{1}{\mu\omega}[\vec{k}(\vec{E}.\vec{E}) - \vec{E}(\vec{E}.\vec{k})] = \frac{1}{\mu\omega}E^2\vec{k} = \frac{E^2}{Z\,k}\vec{k} = \frac{E^2}{Z}\hat{n}$$

Thus the energy flow in the direction of where propagation. Since $\vec{E}$ is normal to $\vec{k}$, then we can write from equation(iv) $kE = \mu\omega H$ or, $E = \frac{\mu\omega}{k} H$ or, $E = \sqrt{\frac{\mu}{\epsilon}} H$ or, $\sqrt{\epsilon}\, E = \sqrt{\mu} H$ or, $\frac{1}{2} \epsilon E^2 = \frac{1}{2}\mu H^2$

This shows that in case of electromagnetic wave in an isotropic dielectric medium, energy is equally shared between electric and magnetic fields. Hence total electromagnetic energy density $u = \frac{1}{2} \epsilon E^2 + \frac{1}{2}\mu H^2 = \epsilon E^2 = \mu H^2$

or, $E^2 = \frac{u}{\epsilon}$ Then Poynting's vector can be written as $\vec{s} = \frac{\frac{u}{\epsilon}}{Z}\hat{n} = \frac{u}{\epsilon\sqrt{\frac{\mu}{\epsilon}}}\hat{n} = u\sqrt{\frac{1}{\mu\epsilon}}\,\hat{n} = uv\,\hat{n}$

Time average of Poynting's vector:

So far we have freely used the complex solution of the field vectors $\vec{E}$ & $\vec{H}$ **with the understanding that actual quantities are given by real parts of the complex solutions. As** $\vec{s}$ is non-linear in the fields it is essential to take a real parts of the fields before multiplying them.

Thus the real point in vector $\vec{s} = \mathrm{Re}\vec{E} \times \mathrm{Re}\vec{H}$ where $\mathrm{Re}\vec{E} = \frac{1}{2}\left(\vec{E} + \vec{E}^*\right)$ and $\mathrm{Re}\vec{H} = \frac{1}{2}\left(\vec{H} + \vec{H}^*\right)$

$$\vec{s} = \frac{1}{4}\left[\left(\vec{E} + \vec{E}^*\right) \times \left(\vec{H} + \vec{H}^*\right)\right] = \frac{1}{4}\left[\vec{E} \times \vec{H} + \vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} + \vec{E}^* \times \vec{H}^*\right]$$

$$\langle\vec{s}\rangle = \frac{1}{4}\left[\langle\vec{E} \times \vec{H}\rangle + \langle\vec{E} \times \vec{H}^*\rangle + \langle\vec{E}^* \times \vec{H}\rangle + \langle\vec{E}^* \times \vec{H}^*\rangle\right] = \frac{1}{4}\left[\langle\vec{E} \times \vec{H}^*\rangle + \langle\vec{E}^* \times \vec{H}\rangle\right]$$

$$\langle\vec{s}\rangle = \frac{1}{4}\left[\left(\vec{E} \times \vec{H}^*\right) + \left(\vec{E}^* \times \vec{H}\right)\right] = \frac{1}{2}Re\left(\vec{E} \times \vec{H}^*\right) = \frac{1}{2}Re\left(\vec{E}^* \times \vec{H}\right)$$

the magnitude of time average of pointing vector is called intensity of radiation(I). Thus the intensity

$$I = |\langle\vec{s}\rangle| = \frac{1}{2}E_0 H_0 = \frac{E_0}{\sqrt{2}}\frac{H_0}{\sqrt{2}} = E_{r.m.s} \times H_{r.m.s}$$

Similarly, the time average of energy density

$$\langle u\rangle = \langle \epsilon E^2\rangle = \langle \epsilon\, \vec{E}.\vec{E}\rangle = \frac{1}{2}\epsilon\, Re\left(\vec{E}.\vec{E}^*\right) = \frac{1}{2}\epsilon\, {E_0}^2 = \epsilon\, {E^2}_{r.m.s}$$

Relation between intensity and time average of energy density

$\frac{I}{\langle u\rangle} = \frac{1}{\epsilon}\frac{H_{r.m.s}}{E_{r.m.s}} = \sqrt{\frac{1}{\mu\epsilon}} = v$ or, $I = v\,\langle u\rangle$

Ordinary laws of reflection and refraction.

Suppose that the XY plane forms the interface Z=0 between two linear homogeneous dielectric (non-conducting) media characterized by permittivities ϵ_1 & ϵ_2 and permeability's μ_1 & μ_2 respectively. Let a plane electromagnetic wave be incident on the interface at the point O . The wave is partly reflected and partly reflected. Let us represent the electric field associated with these waves by $\vec{E} = \overrightarrow{E_0} e^{-j(\vec{k}.\vec{r}-\omega t)}$; $\overrightarrow{E_1} = \overrightarrow{E_{10}} e^{-j(\vec{k}_1.\vec{r}-\omega_1 t)}$; $\overrightarrow{E_2} = \overrightarrow{E_{20}} e^{-j(\vec{k}_2.\vec{r}-\omega_2 t)}$

Where $\overrightarrow{E_0}$, $\overrightarrow{E_{10}}$ & $\overrightarrow{E_{20}}$ are amplitudes $\vec{k}, \vec{k}_1$& $\vec{k}_2$ are propagation vectors ω, ω_1 & ω_2 are the frequencies of the incident, reflected and refracted wave respectively.

Let us now apply the boundary condition that the tangent components of electric field are continuous across the interface Z=0. Thus $\overrightarrow{(E_0)}_t e^{-j(\vec{k}.\vec{r}-\omega t)} + \overrightarrow{(E_{10})}_t e^{-j(\vec{k}_1.\vec{r}-\omega_1 t)} = \overrightarrow{(E_{20})}_t e^{-j(\vec{k}_2.\vec{r}-\omega_2 t)}$

where the subscript t is used to denote tangential component. The condition must be valid for all values of time and it immediately follows that $\omega = \omega_1 = \omega_2$……………(i)

Thus the frequency of electromagnetic wave does not change on reflection and refraction.

The condition (i) must also hold for all points on the interface so we must have $\vec{k}.\vec{r} = \vec{k}_1.\vec{r} = \vec{k}_2.\vec{r}$ at $Z = 0$

therefore $k_x = k_{1x} = k_{2x}$ and $k_y = k_{1y} = k_{2y}$

If the incident beam is assumed to be in the XZ plane then $k_y = 0$ hence $k_{1y} = k_{2y} = 0$ and both $\vec{k}_1$ and $\vec{k}_2$ lie in the XZ plane.

Since Z axis is normal to the interface we may conclude that incident, reflected, refracted waves and normal to the interface at the point of incident all lie in the same plane. From figure

$k_x = k \sin\theta$, $k_{1x} = k_1 \sin\theta_1$ And $k_{2x} = k_2 \sin\theta_2$

There are from the condition $k_x = k_{1x}$ we get $k \sin\theta = k_1 \sin\theta_1$…………(ii)

Since the vectors $\vec{k}$ & $\vec{k}_1$ lie in the same medium we can write for the phase velocity in medium 1

$\frac{\omega}{k} = \frac{\omega_1}{k_1} = \frac{1}{\sqrt{\mu_1\epsilon_1}}$ As $\omega = \omega_1$ then $k = k_1$ hence from equation(ii) $\theta = \theta_1$

That means the angle of incident is equal to the angle of reflection.

Again as $k_{1x} = k_{2x}$ $k_1 \sin\theta_1 = k_2 \sin\theta_2$

$$\frac{k_1}{\omega}\sin\theta_1 = \frac{k_2}{\omega}\sin\theta_2$$

$$\frac{1}{v_1}\sin\theta_1 = \frac{1}{v_2}\sin\theta_2$$

$$\frac{c}{v_1}\sin\theta_1 = \frac{c}{v_2}\sin\theta_2$$

$n_1 \sin\theta_1 = n_2 \sin\theta_2$ Which is well known Snell's law of refraction.

you may proceed another way

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{k_2}{k_1} = \frac{\frac{\omega}{k_1}}{\frac{\omega}{k_2}} = \frac{v_1}{v_2} = \frac{\frac{c}{v_2}}{\frac{c}{v_1}} = \frac{n_2}{n_1} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Fresnel's equations:

The equations which relate the amplitudes of the reflected and refracted waves with that of incident wave are known as Fresnel's equations.

We know that $\vec{E}$vectors in a plane electromagnetic wave is perpendicular to the direction of propagation of the wave. However, it may lie in arbitrary directions in the plane normal to the direction of propagation. It is convenient to consider two extreme cases separately.

1. For which the $\vec{E}$ vector on the incident wave is perpendicular to the plane of incident ; this is called s polarization.
   (s for German word $senkrecht$ meaning perpendicular).
2. For which the $\vec{E}$ vector on the incident wave is parallel to the plane of incident (the plane defined by $\vec{k}$ & $\hat{n}$); this is called $p-$polarization (P for parallel).

The general case of arbitrary polarization can be obtained by a suitable linear combination of the two extreme results.

Case 1: $\vec{E}$ vector on the incident wave is perpendicular to the plane of incident

suppose the $\vec{E}$ vector of the incident wave lies perpendicular to the plane of incidence XZ plane as shown in figure as the media on two sides of interface Z=0 isotropic the $\vec{E}$ vector reflected and refracted waves will also be perpendicular to the plane of incident. Assume that $\vec{E}$ vectors are directed normally out of plane of the paper. $\vec{H}$,$\vec{H_1}$, $\vec{H_2}$ Lie in the plane of paper.

Let us represent the incident, reflected and refracted waves by

$$\vec{E} = \vec{E_0} e^{-j(\vec{k}.\vec{r}-\omega t)} \quad , \quad \vec{H} = \frac{\vec{k}\times\vec{E}}{\omega\mu_1}$$

$$\vec{E_1} = \vec{E_{10}} e^{-j(\vec{k}_1.\vec{r}-\omega_1 t)} \quad , \quad \vec{H_1} = \frac{\vec{k_1}\times\vec{E}_1}{\omega\mu_1}$$

$$\vec{E_2} = \vec{E_{20}} e^{-j(\vec{k}_2.\vec{r}-\omega_2 t)} \quad , \quad \vec{H_2} = \frac{\vec{k_2}\times\vec{E}_2}{\omega\mu_2}$$

Since the electric vectors are all parallel to the interface z=0 , the continuity of the tangent component of$\vec{E}$ field gives us

$$E_0 e^{-j(\vec{k}.\vec{r}-\omega t)} + E_{10} e^{-j(\vec{k}_1.\vec{r}-\omega_1 t)} = E_{20} e^{-j(\vec{k}_2.\vec{r}-\omega_2 t)}$$

The condition must be valid for all values of t and for all points on the interface. This lead to the cancellations of exponential factors. Hence $E_0 + E_{10} = E_{20}$

The continuity of tangential components of$\vec{H}$ fields across the interface z=0 gives

$$-H_0 \cos\theta + H_{10}\cos\theta_1 = -H_{20}\cos\theta_2$$

Now $\vec{H} = \frac{\vec{k}\times\vec{E}}{\omega\mu_1} = \sqrt{\frac{\epsilon_1}{\mu_1}}\frac{\vec{k}\times\vec{E}}{k}$ that means $H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}}E_0$

similarly $H_{10} = \sqrt{\frac{\epsilon_1}{\mu_1}}E_{10}$ and $H_{20} = \sqrt{\frac{\epsilon_2}{\mu_2}}E_{20}$

Therefore $-\sqrt{\frac{\epsilon_1}{\mu_1}}E_0\cos\theta + \sqrt{\frac{\epsilon_1}{\mu_1}}E_{10}\cos\theta_1 = -\sqrt{\frac{\epsilon_2}{\mu_2}}E_{20}\cos\theta_2$

$$\text{Or,} \quad (-E_0 + E_{10})\sqrt{\frac{\epsilon_1}{\mu_1}}E_0\cos\theta = -\sqrt{\frac{\epsilon_2}{\mu_2}}E_{20}\cos\theta_2 \quad \text{as } \theta = \theta_1$$

Then $(-E_0 + E_{10})a = -bE_{20}$ ……………………..(i) $\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta = a$ $\sqrt{\frac{\epsilon_2}{\mu_2}}\cos\theta_2 = b$

$E_0 + E_{10} = E_{20}$……………………………..(ii)

Multiplying equation (ii) by $-a$

$E_0(-2a) = (-b-a)\,E_{20}$ or, amplitude of transmission coefficient $t_\perp = \frac{E_{20}}{E_0} = \frac{2a}{a+b} = \frac{2\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta}{\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta + \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\theta_2}$

Multiplying equation (ii) by b

$E_0(b-a) + (b+a)E_{10} = 0$ or, amplitude of reflection coefficient $r_\perp = \frac{E_{10}}{E_0} = \frac{a-b}{a+b} = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta - \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\theta_2}{\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta + \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\theta_2}$

As most of the dialectics are essentially nonmagnetic we can write $\mu_1 = \mu_2 \approx \mu_0$ again $n_1 = \sqrt{\frac{\epsilon_1}{\epsilon_0}}$ $n_2 = \sqrt{\frac{\epsilon_2}{\epsilon_0}}$

Where n_1, n_2 are the refractive index of these two medium respectively. Then we have

$r_\perp = \frac{E_{10}}{E_0} = \frac{n_1\cos\theta - n_2\cos\theta_2}{n_1\cos\theta + n_2\cos\theta_2}$ $t_\perp = \frac{E_{20}}{E_0} = \frac{2\,n_1\cos\theta}{n_1\cos\theta + n_2\cos\theta_2}$

Now as from Snell's law $n_1\sin\theta = n_2\sin\theta_2$

Fresnel's equation can be rewritten as

$$r_\perp = \frac{E_{10}}{E_0} = \frac{\frac{n_2\sin\theta_2}{\sin\theta}\cos\theta - n_2\cos\theta_2}{\frac{n_2\sin\theta_2}{\sin\theta}\cos\theta + n_2\cos\theta_2} = -\frac{\sin(\theta-\theta_2)}{\sin(\theta+\theta_2)}$$

$$t_\perp = \frac{E_{20}}{E_0} = \frac{2\frac{n_2\sin\theta_2}{\sin\theta}\cos\theta}{\frac{n_2\sin\theta_2}{\sin\theta}\cos\theta + n_2\cos\theta_2} = \frac{2\cos\theta\sin\theta_2}{\sin(\theta+\theta_2)}$$

Case 2: $\vec{E}$ vector on the incident wave is parallel to the plane of incident

suppose the $\vec{E}$ vector of the incident wave lies parallel to the plane of incidence XZ plane as shown in figure as the media on two sides of interface Z=0 isotropic the $\vec{E}$ vector reflected and refracted waves will also be in the plane of incident. Assume that $\vec{E}$ vectors are in the plane of the paper. $\vec{H}$ $\vec{H_1}$ $\vec{H_2}$ Are perpendicular to the plane of incident or a plane of paper.

Let us represent the incident, reflected and refracted waves by

$\vec{E} = \vec{E_0}e^{-j(\vec{k}.\vec{r}-\omega t)}$, $\vec{H} = \frac{\vec{k}\times\vec{E}}{\omega\mu_1}$

$\vec{E_1} = \vec{E_{10}}e^{-j(\vec{k}_1.\vec{r}-\omega_1 t)}$, $\vec{H_1} = \frac{\vec{k_1}\times\vec{E}_1}{\omega\mu_1}$

$\vec{E_2} = \vec{E_{20}}e^{-j(\vec{k}_2.\vec{r}-\omega_2 t)}$, $\vec{H_2} = \frac{\vec{k_2}\times\vec{E}_2}{\omega\mu_2}$

The continuity of tangential components of $\vec{E}$ fields across the interface z=0 gives

$$E_0\cos\theta - E_{10}\cos\theta = E_{20}\cos\theta_2$$

The continuity of tangential components of $\vec{H}$ fields across the interface z=0 gives

$$H_0 + H_{10} = H_{20}$$

As before $H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_0$, $H_{10} = \sqrt{\frac{\epsilon_1}{\mu_1}} E_{10}$, $H_{20} = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{20}$

therefore $\sqrt{\frac{\epsilon_1}{\mu_1}}(E_0 + E_{10}) = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{20}$………………………………..(I) × $\cos\theta$

$$E_0 \cos\theta - E_{10} \cos\theta = E_{20} \cos\theta_2 \dots\dots\dots\dots\dots\dots\dots\dots.(II) \times \sqrt{\frac{\epsilon_1}{\mu_1}}$$

$E_0\, 2\sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta = \left(\sqrt{\frac{\epsilon_2}{\mu_2}} \cos\theta + \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta_2\right) E_{20}$ ∴ amplitude of transmission coefficient $t_{\parallel} = \frac{E_{20}}{E_0} = \frac{2\sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta}{\sqrt{\frac{\epsilon_2}{\mu_2}} \cos\theta + \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta_2}$

$\sqrt{\frac{\epsilon_1}{\mu_1}}(E_0 + E_{10}) = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{20}$………………………………..(I) × $\cos\theta_2$

$E_0 \cos\theta - E_{10} \cos\theta = E_{20} \cos\theta_2$……………………(II) × $-\sqrt{\frac{\epsilon_2}{\mu_2}}$

$$\left(\sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta_2 - \sqrt{\frac{\epsilon_2}{\mu_2}} \cos\theta\right) E_0 + \left(\sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta_2 + \sqrt{\frac{\epsilon_2}{\mu_2}} \cos\theta\right) E_{10} = 0$$

∴ amplitude of reflection coefficient $r_{\parallel} = \frac{E_{10}}{E_0} = \frac{\left(\sqrt{\frac{\epsilon_2}{\mu_2}} \cos\theta - \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta_2\right)}{\sqrt{\frac{\epsilon_2}{\mu_2}} \cos\theta + \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta_2}$

As most of the dialectics are essentially nonmagnetic we can write$\mu_1 = \mu_2 \approx \mu_0$ $n_1 = \sqrt{\frac{\epsilon_1}{\epsilon_0}}$ and $n_2 = \sqrt{\frac{\epsilon_2}{\epsilon_0}}$

Where n_1, n_2 are the refractive index of these two medium respectively. Then we have

$r_{\parallel} = \frac{E_{10}}{E_0} = \frac{n_2 \cos\theta - n_1 \cos\theta_2}{n_2 \cos\theta + n_1 \cos\theta_2}$ $t_{\parallel} = \frac{E_{20}}{E_0} = \frac{2\, n_1 \cos\theta}{n_2 \cos\theta + n_1 \cos\theta_2}$

Now as from Snell's law $n_1 \sin\theta = n_2 \sin\theta_2$

Fresnel's equation can be rewritten as

$$r_{\parallel} = \frac{E_{10}}{E_0} = \frac{n_2 \cos\theta - \frac{n_2 \sin\theta_2}{\sin\theta} \cos\theta_2}{n_2 \cos\theta + \frac{n_2 \sin\theta_2}{\sin\theta} \cos\theta_2} = \frac{\sin(2\theta) - \sin(2\theta_2)}{\sin(2\theta) + \sin(2\theta_2)} = \frac{2\cos(\theta+\theta_2).\sin(\theta-\theta_2)}{2\cos(\theta-\theta_2).\sin(\theta+\theta_2)} = \frac{\tan(\theta-\theta_2)}{\tan(\theta+\theta_2)}$$

$$t_{\parallel} = \frac{E_{20}}{E_0} = \frac{2\frac{n_2 \sin\theta_2}{\sin\theta} \cos\theta}{n_2 \cos\theta + \frac{n_2 \sin\theta_2}{\sin\theta} \cos\theta_2} = \frac{4\cos\theta \sin\theta_2}{\sin(2\theta) + \sin(2\theta_2)} = \frac{4\cos\theta \sin\theta_2}{2\cos(\theta-\theta_2).\sin(\theta+\theta_2)} = \frac{2\cos\theta \sin\theta_2}{\cos(\theta-\theta_2).\sin(\theta+\theta_2)}$$

Physical implications of Fresnel's equation

various important aspects of electromagnetic waves can be described by using Fresnel's equation. In particular, we are interested in polarization effects, phase changes, total internal reflection, reflectance and transmittance.

Polarisation by reflection

we find that $r_{\parallel} = 0$ for $\theta + \theta_2 = \frac{\pi}{2}$, that means that if $\theta + \theta_2 = \frac{\pi}{2}$ then electric field polarised parallel to the plane of incident is not reflected at all. In this condition $r_{\perp} \neq 0$ that means electric field polarised normal to plane of incident is partly reflected. Thus, an un-polarised light consisting of both types of $\vec{E}$ fields incident at an angle $\theta = \theta_P$ satisfying the condition $\theta_P + \theta_2 = \frac{\pi}{2}$ we'll be plane polarised normal to the plane of incidence. This angle θ_P for which $r_{\parallel} = 0$ known as Brewster angle or angle of polarisation. Snell's law in this case gives us

$$\frac{n_2}{n_1} = \frac{\sin\theta}{\sin\theta_2} = \frac{\sin\theta_P}{\sin(\frac{\pi}{2}-\theta_P)} = \tan\theta_P.$$

Thus, the tangent of the angle of polarisation is equal to the refractive index of the reflecting media related to the incident medium. This is known as Brewster law.

<u>Amplitude coefficients and phase change</u>

Fresnel's equation show that amplitude coefficients depend on the value of incident angle θ. Let us to examine the dependence over that in the range of values of θ. For nearly normal incidence θ and θ_2 tends to zero.

$$[r_{\perp}]_{\theta=0} = [r_{\parallel}]_{\theta=0} = \frac{n_1 - n_2}{n_1 + n_2}$$

$$[t_{\perp}]_{\theta=0} = [t_{\parallel}]_{\theta=0} = \frac{2n_1}{n_1 + n_2}$$

As an example for air-glass interface $n_1 = 1$ $n_2 = 1.5$ $[r_{\perp}]_{\theta=0} = [r_{\parallel}]_{\theta=0} = -0.2$

$$[t_{\perp}]_{\theta=0} = [t_{\parallel}]_{\theta=0} = 0.8$$

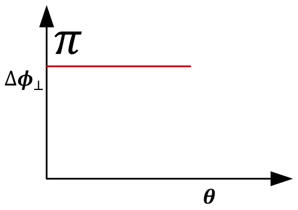
for normal incidences the plane of incidence becomes undefined and any distinction between parallel and perpendicular components vanishes. The difference in signs of $r_{\parallel}$& $r_{\perp}$ arises only because $\vec{E}$ and $\vec{E_1}$ are antiparallel when θ tends to zero, where as $\vec{E}$ and $\vec{E_1}$ Point in the same direction for grazing incident when θ tends to 90° then

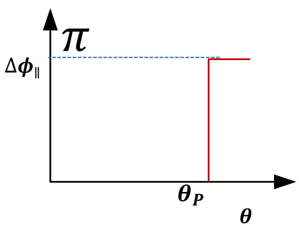
$$r_{\parallel} = r_{\perp} = -1 \qquad t_{\parallel} = t_{\perp} = -0$$

For $n_2 > n_1$ from Snell's law $\theta_2 < \theta$ and hence from $r_{\perp} = -\frac{\sin(\theta-\theta_2)}{\sin(\theta+\theta_2)}$ $r_{\perp}$ is negative for all values of θ . The significance of negative sign is that the component of electric field perpendicular to the plane of incident suffers a phase change $\Delta\phi_{\perp} = \pi$ when reflected from a surface back by an optically denser medium. In contrast from $r_{\parallel} = \frac{\tan(\theta-\theta_2)}{\tan(\theta+\theta_2)}$

Shows that for $n_2 > n_1$ that is $\theta > \theta_2$, $r_{\parallel}$ starts out the positive at θ= 0 and gradually decreases until it becomes zero at the polarizing angle $\theta = \theta_P$. As θ increases beyond θ_P $r_{\parallel}$ becomes progressively more negative and reaches the value -1 at $\theta = 90°$. Thus for $n_2 > n_1$ the electric field polarised parallel to the plane of incident $r_{\parallel}$ is positive that is no phase change $\Delta\phi_{\parallel} = 0$ for $\theta < \theta_P$. And $r_{\parallel}$ is negative that means phase change $\Delta\phi_{\parallel} = \pi$ for $\theta < \theta_P$.

$t_{\parallel}$ and $t_{\perp}$ are always positive that means that refracted waves do not suffer any phase change





Here it is important to mention that we have chosen $\vec{E}, \vec{H}$, $\vec{k}$ vectors to have the same relative orientation when looking down the propagation vector. The $\vec{E}$ field normal to the plane of incident are said to be in phase, if parallel;

π-out of phase, if antiparallel. The $\vec{E}$field parallel to the plane of incident is said to be in phase if their z-components are parallel, and said to be out of phase if z-components are anti-parallel. Note that a phase difference between two

$\vec{E}$fields at the same as that between two associated $\vec{H}$fields. So we need only to look at the vectors normal to the plane of incidence , way that they be $\vec{E}\ or\ \vec{H}$ to determine the relative phase of accompanying fields in the plane of incidence.

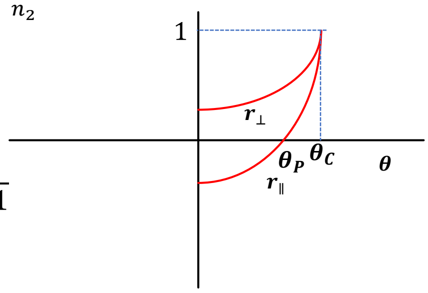
<u>Total internal reflection.</u>

Suppose we consider the case of internal reflection which the incident medium is more dense. i.e., $n_1 > n_2$.

In this case $\theta_2 > \theta$ and equation $r_\perp = -\frac{\sin(\theta-\theta_2)}{\sin(\theta+\theta_2)}$ $r_\perp$ is positive for all values of θ. Hence $r_\perp$ starts from the value $\frac{n_1-n_2}{n_1+n_2}$ At $\theta = 0$ and then increases with θ and attains the value +1 for $\theta = \theta_c$ where θ_c is called critical angle. $\theta_c$is the value of the angle of incidence for which the angle of refraction $\theta_2 = 90°$.

In contrast from $r_\parallel = \frac{\tan(\theta-\theta_2)}{\tan(\theta+\theta_2)}$ Shows that for $n_1 > n_2$ that is $\theta < \theta_2$, $r_\parallel$ starts from a negative value $-\frac{n_1-n_2}{n_1+n_2}$ At $\theta = 0$ and then increases with θ becomes zero at polarising angle $\theta + \theta_P$ and attains the value +1 for $\theta = \theta_c$.

The critical angle is given by Snell's law $\frac{n_2}{n_1} = \frac{\sin\theta_c}{\sin\frac{\pi}{2}}$ $\qquad n\sin\theta_c = 1 \qquad$ where $n = \frac{n_1}{n_2}$



For any angle of incidence θ, $\frac{n_2}{n_1} = \frac{\sin\theta}{\sin\theta_2}$ Then $n\sin\theta = \sin\theta_2$

Hence $\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-n^2\sin^2\theta}$

When θ> θ_c $\quad n\sin\theta_c > 1 \quad$ $\cos\theta_2$ Becomes imaginary $\quad \cos\theta_2 = j\sqrt{n^2\sin^2\theta - 1}$

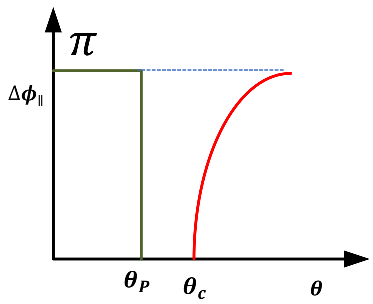
We know examine Fresnel's equation for θ> θ_c using the values of $\sin\theta_2$, $\cos\theta_2$

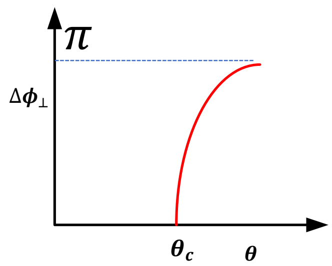
$$r_\perp = \frac{E_{10}}{E_0} = \frac{\text{n}\ \cos\theta-\cos\theta_2}{\text{n}\ \cos\theta+\cos\theta_2} = \frac{\text{n}\ \cos\theta - j\sqrt{n^2\sin^2\theta-1}}{\text{n}\ \cos\theta + j\sqrt{n^2\sin^2\theta-1}}$$

obviously the magnitude of the amplitude reflection coefficient $|r_\perp| = 1$. $r_\perp$is complex with implies a change of phase on reflection. This change of phase depends on the angle of incident greater than θ_c but magnitude of $r_\perp$remains 1.

Similarly we can show that for θ> θ_c the coefficient $r_\parallel$ is complex and$|r_\parallel| = 1$. Again phase change depends on the angle of incidence greater than $\theta_c$but magnitude of $r_\perp$remains 1. Thus both the component of electric field is totally reflected back into the denser medium. This is well-known phenomenon of total internal reflection. It plays an important role in fibre optics.

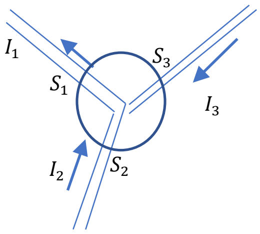
<u>Phase change in internal reflection</u>.





Maxwell's equation and circuit equations

Kirchhoff's current law: considered that several currents meet at a junction in an electric circuit emerging a closed surface S enclosing the junction. If V is the volume bounded by the surface S, then by Gausses divergences theorem. $\iiint_V \vec{\nabla}.\vec{J}\,dV = \oiint_S \vec{J}\,.\hat{n}ds = 0 \quad$ as $\vec{\nabla}.\vec{J} = 0$



Then $\iint_{S_1} \vec{J}\,.\hat{n}dS + \iint_{S_2} \vec{J}\,.\hat{n}dS + \iint_{S_3} \vec{J}\,.\hat{n}dS = 0$

$I_1 - I_2 - I_3 = 0$

Kirchhoff's Voltage law: