3. EM wave in conducting medium

Wave equation in conducting medium, reflection and transmission at metallic surface-skin effect and skin depth, propagation of EM waves between parallel and conducting plates-wave guides (rectangular only).

let us consider a linear, homogeneous and isotropic conducting medium characterised by constant permittivity ε, permeability μ and conductivity σ for simplicity we take the medium is charged free ($\rho = 0$) and external current free such that the current existing in the medium and introduced only by the electromagnetic wave itself. Then $\vec{J} = \sigma\vec{E}$

For such a medium Maxwell's equations take the following form

1. $\vec{\nabla}.\vec{E} = 0$
2. $\vec{\nabla}.\vec{H} = 0$
3. $(\vec{\nabla} \times \vec{E}) = -\mu\frac{\partial\vec{H}}{\partial t}$
4. $(\vec{\nabla} \times \vec{H}) = \sigma\vec{E} + \epsilon\frac{\partial}{\partial t}\vec{E}$

Taking curl of equation $(\vec{\nabla} \times \vec{E}) = -\mu\frac{\partial\vec{H}}{\partial t}$ we get $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$

Or, $\vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E} = \mu\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$

Or, $-\nabla^2\vec{E} = \mu\frac{\partial}{\partial t}\left(\sigma\vec{E} + \epsilon\frac{\partial}{\partial t}\vec{E}\right)$

Or, $\nabla^2\vec{E} - \mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2}{\partial t^2}\vec{E} = \vec{0}$……………………..(i)

Similarly Taking curl of equation$(\vec{\nabla} \times \vec{H}) = \sigma\vec{E} + \epsilon\frac{\partial}{\partial t}\vec{E}$we get $\nabla^2\vec{H} - \mu\sigma\frac{\partial\vec{H}}{\partial t} - \mu\epsilon\frac{\partial^2}{\partial t^2}\vec{H} = \vec{0}$……………(ii)

suppose we are interested in the plane wave solutions and assume that

$\vec{E}(\vec{r},t) = \overrightarrow{E_0}e^{j(\vec{k}.\vec{r}-\omega t)}$, $\vec{H}(\vec{r},t) = \overrightarrow{H_0}e^{j(\vec{k}.\vec{r}-\omega t)}$ [Where $\vec{E}_0, \vec{H}_0$ are complex amplitudes. $\vec{k} = k\hat{n}$ propagation vector]

substituting the solution in equation (i) and (ii) we get $[-k^2 + j\omega\sigma\mu + \omega^2\mu\epsilon]\vec{E} = \vec{0}$

or, $-k^2 + j\omega\sigma\mu + \omega^2\mu\epsilon = 0$

or, $k^2 = \omega^2\mu\epsilon + j\omega\sigma\mu$ k is thus complex

let $k = \alpha + j\beta$ then $k^2 = \alpha^2 - \beta^2 + j2\alpha\beta$ then $\boldsymbol{\alpha^2 - \beta^2 = \omega^2\mu\epsilon}$………(iii) & $2\alpha\beta$ $=\omega\sigma\mu$……… And(iv)

Then $(\alpha^2 + \beta^2)^2 = (\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2 = (\omega^2\mu\epsilon)^2 + (\omega\sigma\mu)^2 = \omega^4\mu^2\epsilon^2\left(1 + \frac{\sigma^2}{\epsilon^2\omega^2}\right)$

Or, $\boldsymbol{\alpha^2 + \beta^2 = \omega^2\mu\epsilon\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}}}$………….(v)

from equation(iii) and (v)

$2\alpha^2 = \omega^2\mu\epsilon\left[\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} + 1\right]$ then $\boldsymbol{\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} + 1\right]^{\frac{1}{2}}}$

and $2\beta^2 = \omega^2\mu\epsilon\left[\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} - 1\right]$ then $\boldsymbol{\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} - 1\right]^{\frac{1}{2}}}$

Then we have

$\vec{E}(\vec{r},t) = \overrightarrow{E_0}e^{-j((\alpha+j\beta)\hat{n}.\vec{r}-\omega t)} = \overrightarrow{E_0}e^{-\beta\hat{n}.\vec{r}}e^{-j(\alpha\hat{n}.\vec{r}-\omega t)}$ & $\vec{H}(\vec{r},t) = \overrightarrow{H_0}e^{-j((\alpha+j\beta)\hat{n}.\vec{r}-\omega t)} = \overrightarrow{H_0}e^{-\beta\hat{n}.\vec{r}}e^{-j(\alpha\hat{n}.\vec{r}-\omega t)}$

Solution show that the field amplitudes are specially attenuated. The physical reason of the attenuation is that the wave sets up electric current in the medium which causes dissipation of energy in form of Joule heating.

The quantity β is the imaginary part of wave number k, is a measure of attenuation and is called attenuation constant. It depends on the frequency ω and the conductivity σ.

For non-conductors σ=0 then $\beta = 0$ hence there is no attenuation of field vectors.

For good conductor is $\frac{\sigma}{\omega\epsilon} \gg 1$ and then we have $\alpha \approx \beta \approx \sqrt{\frac{\mu\epsilon\omega}{2}}$

the quantity $\frac{1}{\beta}$ measures is the depth at which electromagnetic wave entering a conductor is attenuated to $\frac{1}{e}$ of its initial amplitude at the surface. It is known as skin depth or penetration depth into a conduct medium.

The skin depth $\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\epsilon\omega}}$ for a good conductor.

Obviously, skin depth decreases with increasing frequency and conductivity. For good conductors at high frequencies skin depth is very small. That is why in high-frequency circuits current flows only through the surface of the good conductors. This phenomenon is called skin effect. Due to this effect the AC resistance of a conductor is greater than that this resistance. For this in high-frequency circuits it is better to use in number of fine standard wires instead of a thick way. It increases surface area for a given cross-sectional area and hence reduce the resistance.

At microwave frequencies skin depth is very small for Ag ($\sim 10^{-3} mm$). As a result, in the microwave region the performance of the waveguide made of pure Ag and other waveguide made of Ag-coated brass would appear identical. This technique is used to reduce the material cost of good conductors.

Note that the attenuation of electromagnetic wave in conducting sea-water creates problem in radio - communications with submerged submarine.

Wave length, propagation speed and index of refraction.

The real part of k, that is α the deadline is the wavelength, propagation speed of the wave and index of refraction

$\lambda_c = \frac{2\pi}{\alpha}$; $v = \frac{\omega}{\alpha}$; $n = \frac{c\alpha}{\omega}$.

For good conductors $\alpha = \sqrt{\frac{\mu\epsilon\omega}{2}}$ then $\lambda_c = 2\pi\sqrt{\frac{2}{\mu\epsilon\omega}}$; $v = \omega\sqrt{\frac{2}{\mu\epsilon\omega}} = \sqrt{\frac{2\omega}{\mu\epsilon}}$; $n = \frac{c}{\omega}\sqrt{\frac{\mu\epsilon\omega}{2}} = c\sqrt{\frac{\mu\epsilon}{2\omega}}$

note that skin depth can be expressed in terms of wavelength as $\delta = \frac{\lambda_c}{2\pi}$

The relative directions of $\vec{E}, \vec{H}$ & $\vec{k}$

Substituting the solutions $\vec{E}(\vec{r},t) = \vec{E_0}e^{j(\vec{k}.\vec{r}-\omega t)}$, $\vec{H}(\vec{r},t) = \vec{H_0}e^{j(\vec{k}.\vec{r}-\omega t)}$ in equation (i) and (ii) we get

$\vec{k}.\vec{E} = 0$ & $\vec{k}.\vec{H} = 0$ thus both $\vec{E}$ & $\vec{H}$ are perpendicular to $\vec{k}$. This implies that electromagnetic waves are transvers in nature.

Again From $(\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial \vec{H}}{\partial t}$ we get $\boldsymbol{j\vec{k} \times \vec{E} = -\mu(-j\omega\vec{H})}$ **or,** $\boldsymbol{\vec{k} \times \vec{E} = \mu\omega\vec{H}}$**……….(iv)**

And from $(\vec{\nabla} \times \vec{H}) = \sigma\vec{E} + \epsilon\frac{\partial}{\partial t}\vec{E}$ we get $\boldsymbol{j\vec{k} \times \vec{H}} = \sigma\vec{E} + \boldsymbol{\epsilon(-j\omega\vec{E})}$ **or,** $\boldsymbol{\vec{k} \times \vec{H}} = -(\boldsymbol{\omega\epsilon} + j\sigma)\boldsymbol{\vec{E}}$**……………..(v)**

Relative phase of $\vec{E}$&$\vec{H}$

We have from equation (iv) $\vec{H} = \frac{1}{\omega\mu}(\boldsymbol{\vec{k} \times \vec{H}}) = \frac{k}{\omega\mu}(\boldsymbol{\hat{n} \times \vec{H}}) = \frac{\alpha+j\beta}{\omega\mu}(\boldsymbol{\hat{n} \times \vec{H}})$……….(vi)

This equation shows that $\boldsymbol{\vec{E}}$&$\boldsymbol{\vec{H}}$ **and not in phase in a conductor.**

Then we have $\vec{H} = \frac{\sqrt{\alpha^2+\beta^2}}{\omega\mu}(\boldsymbol{\hat{n} \times \vec{E_0}})e^{j[\vec{k}.\vec{r}-(\omega t-\phi)]}$ taking $\alpha + j\beta = \sqrt{\alpha^2+\beta^2}\mathrm{e}^{-j\phi}$; $\phi = \tan^{-1}\frac{\beta}{\alpha}$

$\vec{H}$ lags behind $\vec{E}$ by the phase angle $\phi = \tan^{-1}\frac{\beta}{\alpha} = \frac{1}{2}\tan^{-1}\frac{\sigma}{\omega\epsilon}$

[*as* $k = \alpha + j\beta = \sqrt{\alpha^2+\beta^2}\mathrm{e}^{-j\phi} = (\omega^2\mu\epsilon + j\omega\sigma\mu)^{\frac{1}{2}}$ *then* $(\alpha^2+\beta^2)\mathrm{e}^{-2j\phi} = \omega^2\mu\epsilon + j\omega\sigma\mu = \sqrt{(\omega^2\mu\epsilon)^2 + (\omega\sigma\mu)^2}\mathrm{e}^{-j\phi'}$
where $\phi' = \tan^{-1}\frac{\omega\sigma\mu}{\omega^2\mu\epsilon} = \tan^{-1}\frac{\sigma}{\omega\epsilon}$ hence $\phi = \frac{1}{2}\tan^{-1}\frac{\sigma}{\omega\epsilon}$]

For good conductor it is $\alpha \approx \beta$ and $\phi = 45°$ then phase difference between $\vec{H}$ & $\vec{E}$ for a perfect conductor is 45°
the relative magnitudes of $\vec{H}$ & $\vec{E}$

$$\frac{|\vec{H}|}{|\vec{E}|} = \frac{H_0}{E_0} = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} = \sqrt{\frac{\epsilon}{\mu}}\left[1 + \frac{\sigma^2}{\epsilon^2\omega^2}\right]^{\frac{1}{4}}$$

For good conductors $\frac{|\vec{H}|}{|\vec{E}|} = \sqrt{\frac{\sigma}{\omega\mu}}$ Thus we have $|\vec{H}| \gg |\vec{E}|$

That indicates that in a good conducting medium the field energy is not equally shared between electric fields and magnetic fields but is almost entirely magnetic in nature.

Poynting's vector.

The time average Poynting's vector is $\langle\vec{s}\rangle = \frac{1}{2}Re(\vec{E}^* \times \vec{H})$

using equation(vi) we get $\langle\vec{s}\rangle = \frac{1}{2}\frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} Re\left[\vec{E} \times (\hat{n} \times \vec{E^*})\right]\mathrm{e}^{-j\phi}$

$$= \frac{1}{2}\frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} Re\left[\hat{n}(\vec{E}.\vec{E^*}) - \vec{E^*}(\vec{E}.\hat{n})\right]\mathrm{e}^{-j\phi}$$

$\langle\vec{s}\rangle = \frac{1}{2}\frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega}E_0^2 e^{-2\beta\hat{n}.\vec{r}}\cos\phi\ \hat{n}$ [as $\vec{E}.\hat{n} = 0$]

for good conductors $\alpha \approx \beta \approx \sqrt{\frac{\mu\epsilon\omega}{2}}$ and $\phi = 45°$ then $\langle\vec{s}\rangle = \frac{1}{2}\sqrt{\frac{\sigma}{2\mu\omega}}E_0^2 e^{-2\beta\hat{n}.\vec{r}}\ \hat{n}$.

Thus, energy flow is along the direction of propagation of the wave and is damped off exponentially.

Average electric energy density is $\langle u_e\rangle = \frac{1}{2}Re\frac{1}{2}(\vec{E}.\vec{D^*}) = \frac{1}{4}\epsilon Re\,(\vec{E}.\vec{E^*}) = \frac{1}{4}\epsilon E_0^2 e^{-2\beta\hat{n}.\vec{r}}$

Average magnetic energy density is $\langle u_m\rangle = \frac{1}{2}Re\frac{1}{2}(\vec{B}.\vec{H^*}) = \frac{1}{4}\mu Re\,(\vec{H}.\vec{H^*}) = \frac{1}{4}\mu H_0^2 e^{-2\beta\hat{n}.\vec{r}}$

Then $\frac{\langle u_m\rangle}{\langle u_e\rangle} = \frac{\mu H_0^2}{\epsilon E_0^2} = \left[1+\frac{\sigma^2}{\epsilon^2\omega^2}\right]^{\frac{1}{2}}$

Obviously for good conductor $\langle u_m\rangle \gg \langle u_e\rangle$That means the field energy is almost entirely magnetic in nature.

Hence electromagnetic energy density

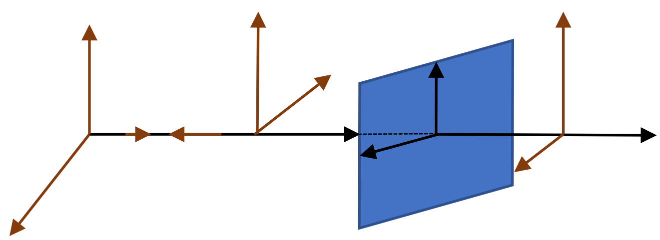
$$\langle u\rangle = \langle u_m\rangle + \langle u_e\rangle = \frac{1}{4}e^{-2\beta\hat{n}.\vec{r}}(\epsilon E_0^2+\mu H_0^2) = \frac{1}{4}e^{-2\beta\hat{n}.\vec{r}}\epsilon E_0^2\left(1+\left[1+\frac{\sigma^2}{\epsilon^2\omega^2}\right]^{\frac{1}{2}}\right)$$

In term of α, the real part of wave number the electromagnetic energy density can be expressed as

$\langle u\rangle = \frac{\alpha^2}{2\mu\omega^2}E_0^2 e^{-2\beta\hat{n}.\vec{r}}$ [as $\boldsymbol{\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1\right]^{\frac{1}{2}}}$]

Reflection and transmission at a conducting surface

Suppose XY-plane forms and interface $(z=0)$ between a dielectric medium (I) characterised by permittivity and permeability ϵ_1 and μ_1 respectively and a conducting medium (II) characterised by permittivity, permeability and conductivity ϵ_2 , μ_2 and σ respectively. Let at plane electromagnetic wave of frequency ω polarised along the X-direction and travelling along Z direction be incident normally on the interface. The incident, reflected and transmitted electric and magnetic field of incident wave are $\vec{E}$ & $\vec{H}$, $\vec{E_1}$ & $\vec{H_1}$ and $\vec{E_2}$ & $\vec{H_2}$ Respectively.



Let us described the wave fields by

Incident wave: $\vec{E} = \hat{\imath}E_0 e^{j(kz-\omega t)}$; $\vec{H} = \hat{\jmath}H_0 e^{j(kz-\omega t)}$

Reflected wave: $\vec{E_1} = \hat{\imath}\,E_{10}e^{j(k_1z-\omega_1 t)}$; $\vec{H_1} = -\hat{\jmath}\,H_{10}e^{j(k_1z-\omega_1 t)}$

Transmitted wave : $\vec{E_2} = \hat{\imath}\,E_{20}e^{j(k_2z-\omega_2 t)}$; $\vec{H_2} = \hat{\jmath}\,H_{20}e^{j(k_2z-\omega_2 t)}$

Where the amplitude of $\vec{H}$-fields are related to the amplitude of $\vec{E}$ – fields as

$H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}}E_0$ $H_{10} = \sqrt{\frac{\epsilon_1}{\mu_1}}E_{10}$ and $H_{20} = \sqrt{\frac{\epsilon_2}{\mu_2}}E_{20}$

the reflected and transmitted waves are assumed to have the same frequency as that of incident wave otherwise the boundary conditions at $z=0$ cannot be satisfied for all values of t . The minus sign in $\vec{H_1}$ Is equipped by the fact that the poynting's vector must aim the direction of propagation.

Now and applying the boundary conditions from the continuity of tangential component of $\vec{E}$ − fields across the boundary we have $E_0 + E_{10} = E_{20}$…………(vii)

Similarly, the continuity of $\vec{H}$ − fields across the boundary gives $H_0 - H_{10} = H_{20}$

Or, $\sqrt{\frac{\epsilon_1}{\mu_1}}(E_0 - E_{10}) = \sqrt{\frac{\epsilon_2}{\mu_2}}E_{20}$

Taking the media to be nonmagnetic$\mu_1 = \mu_2 \approx \mu_0$ then $n_1 = \sqrt{\frac{\epsilon_1}{\epsilon_0}}$ and $n_2 = \sqrt{\frac{\epsilon_2}{\epsilon_0}}$

$n_1(E_0 - E_{10}) = n_2 E_{20}$………(viii) from equation(vii) & (viii) we get

$r_{12} = \frac{E_{10}}{E_0} = \frac{n_1 - n_2}{n_1 + n_2}$ $\quad$ $t_{12} = \frac{E_{20}}{E_0} = \frac{2n_1}{n_1 + n_2}$

Where r_{12} is the amplitude reflection coefficient and t_{12} *is* amplitude transmission coefficient.

Note that these results are formally identical for in the case of interface between to di-electric media. But in this case inside the metallic media the refractive index n_2 is complex quantity $n_2 = \frac{c}{\frac{\omega}{k_2}} = \frac{c}{\omega}(\alpha + j\beta)$

with $\varepsilon = \varepsilon_2$ $\quad$ $\mu = \mu_2 \approx \mu_0$ $\quad$ $n_2 = n_r + jn_i$

For good conductor arts $\sigma \gg \omega\varepsilon_2$ then $\alpha \approx \beta \approx \sqrt{\frac{\mu_0 \epsilon \omega}{2}}$ and $n_2 = \frac{c}{\omega}\sqrt{\frac{\mu_0 \epsilon \omega}{2}}(1+j) = \sqrt{\frac{\sigma}{2\varepsilon_0 \omega}}\,(1+j)$

Hence in this case $n_r = n_i = \sqrt{\frac{\sigma}{2\varepsilon_0 \omega}}$

Now $r_{12} = \frac{E_{10}}{E_0} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{n_1 - (n_r + jn_i)}{n_1 + (n_r + jn_i)} = -\frac{(n_r - n_1) + jn_i}{(n_r + n_1) + jn_i}$ $\quad$ $t_{12} = \frac{E_{20}}{E_0} = \frac{2n_1}{n_1 + n_2} = \frac{2n_1}{(n_r + n_1) + jn_i}$

The complex nature of r_{12} & t_{12} indicates that the reflected and transmitted waves may be phase shifted relative to the incident wave. We can write the complex quantities r_{12} & t_{12} as $r_{12} = |r_{12}|e^{-i\phi_1}$ And $t_{12} = |t_{12}|e^{-i\phi_2}$

Where $|r_{12}| = \left[\frac{(n_r - n_1)^2 + n_i^2}{(n_r + n_1)^2 + n_i^2}\right]^{\frac{1}{2}}$ And $|t_{12}| = \frac{2n_1}{[(n_r + n_1)^2 + n_i^2]^{\frac{1}{2}}}$

Then $\phi_1 = \tan^{-1}\frac{n_i}{(n_r - n_1)} - \tan^{-1}\frac{n_i}{(n_r + n_1)} = \tan^{-1}\frac{2n_1 n_i}{n_r^2 - n_1^2 + n_i^2}$ And $\phi_2 = -\tan^{-1}\frac{n_i}{(n_r + n_1)}$

The results show that the reflected and transmitted $\vec{E}$ fields are phase shifted relative to the incident $\vec{E}$ fields.

For perfect conduct $\sigma = \infty$ $\quad n_r = n_i = \infty$ and hence $r_{12} = -1$. It indicates a phase shift by $180°$ due to reflection from a perfect conducting surface. Show the component of $\vec{E}$ and $\overrightarrow{E_1}$ Cancel at the surface of perfect conductor. This is expected because fields in perfect conductor must vanish and they are can be no transmitted wave($\overrightarrow{E_2} = \vec{0}$).

The totally reflected wave combines with the incident wave and forms is standing of stationary wave with a node on the surface. The standing wave is given by $\overrightarrow{E_s} = \vec{E} + \overrightarrow{E_1} = \hat{\imath}E_0\left[e^{j(kz-\omega t)} - e^{-j(kz-\omega t)}\right] = \hat{\imath}2E_0(j\sin kz)e^{-j\omega t}$

Taking real part we can write $\overrightarrow{E_s} = \hat{\imath}2E_0(j\sin kz)\sin\omega t$.

Now the reflectance of the metal surfaces is $R = r_{12}.r_{12}^* = |r_{12}|^2 = \frac{(n_r - n_1)^2 + n_i^2}{(n_r + n_1)^2 + n_i^2}$

And the transmittance is $T = 1 - \mathrm{R} = 1 - \frac{(n_r - n_1)^2 + n_i^2}{(n_r + n_1)^2 + n_i^2} = \frac{4n_r n_1}{(n_r + n_1)^2 + n_i^2}$

The transmitted wave is eventually absorbed in the conduct medium. For this it is customary to call it absorptance A.

$$T = A = 1 - R$$

For good conductors $n_r = n_i = \sqrt{\frac{\sigma}{2\varepsilon_0 \omega}} \gg 1$

if the incident medium is air then $n_1 = 1$ then for incident from air on a good conducting surface we have

$T = A = \frac{2}{n_i} = 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}}$ $\quad$ $R = 1 - 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}}$............(ix)

This approximated relation is known as **Hagen-Rubens formula.** This relation is found to be valid for metals at frequencies as high as 10^{15} Hz. For a moderately good conductors the relation is valid for frequencies below the microwave frequencies.

For a perfect conduct $\sigma = \infty$ $R = 1$ and $T = 0$ thus, in this case the incident wave is totally reflected. This explain why metals are opaque to light and why excellent conductors, such as silver, make good mirrors.

Let us use the relation(ix) to determine the reflectance of year to silver surface

here conductivity $\sigma = 6 \times 10^7/\Omega - m$ frequency $f = 6.4 \times 10^{14}$ Hz

$$R = 1 - \sqrt{\frac{8 \times 2 \times 3.14 \times 6.4 \times 10^{14} \times 8.854 \times 10^{-12}}{6 \times 10^7}} = 0.9311$$

hence almost 93% of incident energy is reflected and 7% energy is absorbed by the silver.

As already said this high reflectance of seawater level for radio waves causes problems in radio communication from land with submerged submarines.

Propagation of electromagnetic waves between parallel conducting plates

a structured consisting of highly conducting was in the form of a long open pipe

$\vec{E} = \hat{\imath}E'_0 e^{j(\vec{k}.\vec{r}-\omega t)} + \hat{\imath}E'_{10} e^{-j(\vec{k}_1.\vec{r}-\omega_1 t)}$

$$E'_0 = E'_{10}$$

$$\vec{E} = \hat{\imath}E'_0 [e^{j(\vec{k}.\vec{r}-\omega t)} - e^{-j(\vec{k}_1.\vec{r}-\omega_1 t)}]$$

$$\vec{k}.\vec{r} = k\cos\theta\, y + k\sin\theta\ z$$

$\vec{k}_1.\vec{r} = -k\cos\theta\, y + k\sin\theta\ z$

$$\vec{E} = \hat{\imath}E'_0 \left[e^{j(k\cos\theta y)} - e^{-j(k\cos\theta y)}\right]. e^{-j(k\sin\theta z-\omega t)}$$

$$= \hat{\imath}E_0[\sin(ky\cos\theta)]. e^{-j(k\sin\theta z-\omega t)}$$

$$E_0 = 2E'_0$$

$\vec{E}$ $\quad y = a \quad \sin(ka\cos\theta) = 0 \quad\quad ka\cos\theta = n\pi$

$$\vec{E} = \hat{\imath}E_0[\sin(k_c y)]. e^{-j(k_g z-\omega t)}$$

$k\cos\theta = k_c \quad\quad\quad k\sin\theta = k_g$

$$\lambda_g = \frac{2\pi}{k_g} = \frac{2\pi}{k\sin\theta} = \frac{\lambda_0}{\sin\theta}$$

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi}{k\cos\theta} = \frac{\lambda_0}{\cos\theta}$$

$$\frac{1}{{\lambda_g}^2} + \frac{1}{{\lambda_c}^2} = \frac{1}{{\lambda_0}^2}$$

$k_c.a = n\pi \quad \lambda_g = \frac{2\pi}{k_c} = \frac{2a}{n} \quad \lambda_0 < \lambda_c = \frac{2a}{n}$

$$\lambda_g = \frac{1}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

$$v_p = \frac{\omega}{k_g} = \frac{\omega}{k\sin\theta} = \frac{c}{\sin\theta}$$

$\sin\theta \le 1 \quad v_p \ge c \quad \theta = 0 \quad \lambda_0 = \lambda_c \quad v_p \to \infty$

$$v_g = \frac{d\omega}{dk_g}$$

$$k_g{}^2 = k^2 \sin^2\theta = k^2(1 - \cos^2\theta) = k^2 - k_c{}^2 = k^2 - \frac{n^2\pi^2}{a^2}$$

$$k_g{}^2 = \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2}$$

$$2k_g = \frac{2\omega}{c^2}\frac{d\omega}{dk_g}$$

$$\frac{d\omega}{dk_g} = \frac{k_g c^2}{\omega} = \frac{k \sin\theta\, c^2}{ck} = c\sin\theta$$

$$v_g = c\sin\theta$$

$$v_g . v_p = c\sin\theta\, . \frac{c}{\sin\theta} = c^2$$