ELECTRICITY AND MAGNETISM

PHS-A-CC2-3-TH

1. Electrostatic Field.

a) Coulombs law and principle of superposition leading to the definition of electrostatic field. Field lines.
b) Divergence of electrostatic field. Flux, Gausses theorem of electrostatic. Application of Gausses theorem to find the electric field due to charge configurations with spherical, cylindrical and planer symmetry.
c) Curl of the electrostatic field and its conservative nature. Electric potential. Potential for a uniformly charged spherical shell and solid sphere. Calculation of electric field from potential.
d) Laplace's and Poisson equation. Uniqueness theorems. Method of images and its application to (1) plane infinite sheet and (2) sphere.
e) Conductors: electric field and charge density inside and on the surface of a conductor. Force per unit area on the surface. Capacitance of a conductor. Capacitance an isolated spherical conductor. Parallel plate condenser.
f) Electrostatic energy of system of charges. Electrostatic energy of a charged sphere.
g) Energy per unit volume in electrostatic field.

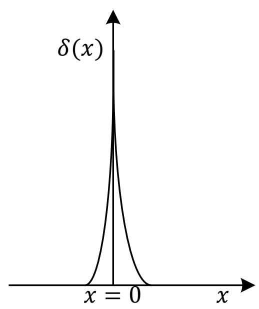
Mathematical background

1. Dirac Delta function

Discrete sources such as point charges, elementary currents, current shells, etc constitute singularities of electromagnetic fields. One convenient method treat these discrete sources is the use of Dirac delta function.

In one dimension Dirac delta function at a point $x = x_0$ is designated by $\delta(x - x_0)$ and at $x = 0$ by $\delta(x)$.

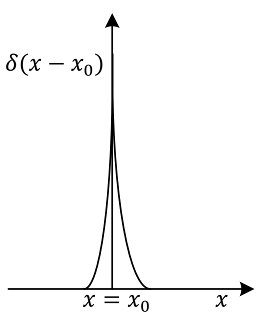
It has the following properties. 1. $\delta(x - x_0) = \begin{cases} 0, & for\ x \neq x_0 \\ \infty & for\ x = x_0 \end{cases}$



2. $\int_a^b \delta(x - x_0)dx = \begin{cases} 1, & if\ x_0\ is\ in\ (a, b) \\ 0 & if\ x_0\ is\ not\ in\ (a, b) \end{cases}$

3. $\int_a^b f(x)\delta(x - x_0)dx = \begin{cases} f(x_0), & if\ x_0\ is\ in\ (a, b) \\ 0 & if\ x_0\ is\ not\ in\ (a, b) \end{cases}$

4. $\int_a^b f(x)\delta'(x - x_0)dx = \begin{cases} -f'(x_0), & if\ x_0\ is\ in\ (a, b) \\ 0 & if\ x_0\ is\ not\ in\ (a, b) \end{cases}$



5. $\delta(\vec{r} - \vec{r}_0) = \begin{cases} 0, & for\ \vec{r} \neq \vec{r}_0 \\ \infty & for\ \vec{r} = \vec{r}_0 \end{cases}$

6. $\iiint_V \delta(\vec{r} - \vec{r}_0)dV = \begin{cases} 1, & if\ \vec{r}_0\ is\ in\ V \\ 0 & if\ \vec{r}_0\ is\ not\ in\ V \end{cases}$

7. $\int_V f(\vec{r})\delta(\vec{r} - \vec{r}_0)dV = \begin{cases} f(\vec{r}_0), & if\ \vec{r}_0\ is\ in\ V \\ 0 & if\ \vec{r}_0\ is\ not\ in\ V \end{cases}$

Note that the Dirac delta function is not continuous one, it can be considered as the limit of an ordinary function. For example $\delta(x) =$ $\boxed{\lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi}\epsilon} e^{-\frac{x^2}{\epsilon^2}}}$

More properties of delta function. $\boxed{\delta(x) = -\delta(-x); \quad \delta'^{(x)} = -\delta'^{(-x)}; \quad \delta(ax) = \frac{\delta(x)}{a}}$

Another important property of Dirac delta function comes from its relation to the divergence or a Laplacian operator

$$\vec{\nabla}.\frac{\vec{r}}{r^3} = \vec{\nabla}.\frac{\vec{r}}{r^3} = \left[\frac{1}{r^3}\vec{\nabla}.\vec{r} + \vec{r}.\vec{\nabla}\left(\frac{1}{r^3}\right)\right] = \left[\frac{3}{r^3} + \vec{r}.(-3r^5\vec{r})\right] = \left[\frac{3}{r^3} - \frac{3}{r^3}\right] = 0 \text{ for } r \neq 0$$

And $\vec{\nabla}.\frac{\vec{r}}{r^3}$ becomes indeterminate as $r \to 0$. Hence to avoid this situation we can represent $\vec{\nabla}.\frac{\vec{r}}{r^3}$ by Dirac Delta function. Applying divergence theorem to a small sphere of radius R around the origin we get

$$\iiint_V \vec{\nabla}.\frac{\vec{r}}{r^3}\, dV = \oiint_S \frac{\vec{r}.\hat{n}dS}{r^3} = \frac{1}{R^2}\oiint_S dS = \frac{1}{R^2}4\pi R^2 = 4\pi = 4\pi \iiint_V \delta(\vec{r})dV = \iiint_V 4\pi\delta(\vec{r})dV$$

since the result is true regardless to how small the radius, one can replace $\vec{\nabla}.\frac{\vec{r}}{r^3}$ by $4\pi\delta(\vec{r})$.

Hence we can write $\boldsymbol{\delta(\vec{r}) = \frac{1}{4\pi}\vec{\nabla}.\frac{\vec{r}}{r^3} = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right)}$

2. Arithmetic and geometric progression

A.P : $S_n = a + (a + d) + (a + 2d) + \cdots \dots . + \{a + (n - 1)d\} = \frac{n}{2}\{2a + (n - 1)d\}$

G.P : $S_n = a + ar + ar^2 + \cdots . + ar^{(n-1)} = \frac{a(1-r^n)}{1-r}$ **[Note that for $\boldsymbol{n = \infty, S_\infty = \frac{a}{1-r}\ for\ |r| \leq 1}$]**

3. Binomial expansion

$$(1 + x)^n = 1 + {}^n_1c\ x + {}^n_2c\ x^2 + \cdots \dots \dots = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^2 + \dots\dots$$

Note that for n is a positive integer the series terminate for all x. but when n is not a positive integer, the series does not terminate. The infinite series is convergent for $|x| < 1$

$$\boldsymbol{(a + b)^n = a^n + {}^n_1c\ a^{n-1}b^1 + {}^n_2c\ a^{n-2}b^2 + \cdots = 1 + na^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots\dots}$$

4. Taylor and McLaurin series:

A **Taylor series** is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function $f(x)$ about a point $x = a$ is given by

$$f(x) = f(a) + \frac{1}{1!}(x - a)f'(a) + \frac{1}{2!}(x - a)^2 f''(x - a) + \cdots \dots . + \frac{1}{n!}(x - a)^n f^{(n)}(x - a)$$

If $a = 0$, the expansion is known as a **Maclaurin series.**

5. Power series with real variables

Powers of natural numbers	
$\sum_{k=1}^{n} k = \frac{1}{2}n(n + 1)$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots\dots = \ln 2$
$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1)$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots . = \frac{\pi}{4}$
$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n + 1)^2$	$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \dots . = \frac{\pi^2}{6}$
	$\sum_{k=1}^{\infty} \frac{1}{k^3} =$

Special power series	
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (for all values of x)	$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots \dots\dots$ (for $-1 < x < 1$)
$a^x = 1 + x\ln a + \frac{(x\ln a)^2}{2!} \dots$ for all values of x	$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} \dots \dots \dots .$ (for $-\frac{\pi}{2} < x < \frac{\pi}{2}$)
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots \dots .$ (for all values of x)	$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} \dots \dots .$ (for $-\frac{\pi}{2} < x < \frac{\pi}{2}$)
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots \dots\dots$ (for all values of x)	$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} \dots \dots \dots \dots$ (for $-1 < x < 1$)
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \dots\dots$ (for all values of x)	$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \dots \dots \dots \dots\dots$ (for $-1 < x < 1$)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \dots \dots$ (for all values of x)	$\tan^{-1} x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \dots \dots \dots \dots$ (for $-1 < x < 1$)

6. Some important vector identity

$\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{B})$

$\vec{\nabla}\times(\vec{A}\times\vec{B}) = \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A}) + (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B}$

$\vec{\nabla}(\vec{A}.\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{B}.\vec{\nabla})\vec{A} + (\vec{A}.\vec{\nabla})\vec{B}$

7. Gauss's divergence law $\iiint_V (\vec{\nabla}\cdot\vec{F})\, dV = \oiint_S \vec{F}\cdot\overrightarrow{dS}$

8. Stoke's law $\iint_S (\vec{\nabla}\times\vec{F}).\overrightarrow{dS} = \oint_C \vec{F}.\overrightarrow{dl}$

9. Green's theorem $\oiint_S \psi\,\vec{\nabla}\phi.\overrightarrow{dS} = \iiint_V (\psi\nabla^2\phi + \vec{\nabla}\psi.\vec{\nabla}\phi)\, dV$

10. Green's 2nd theorem $\iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\, dV = \oiint_S (\psi\vec{\nabla}\phi - \phi\vec{\nabla}\psi).\overrightarrow{dS}$

11. Vector differential operators in different co-ordinate systems

Cylindrical co-ordinates(ρ,φ,z)	Spherical co-ordinate (r,θ,φ)	Remarks
$\vec{\nabla}\phi = \hat{\rho}\frac{\partial\phi}{\partial\rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial\phi}{\partial\varphi} + \hat{k}\frac{\partial\phi}{\partial z}$ $\vec{\nabla}.\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}(A_\varphi) + \frac{\partial}{\partial z}(A_z)$ $\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\varphi} & \hat{k}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\varphi & A_z\end{vmatrix}$ $\nabla^2\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}(\phi) + \frac{\partial^2}{\partial\varphi^2}(\phi)$ **Elementary area** $\rho d\varphi dz$ **(curved surface)** **Elementary volume** $\rho d\rho d\varphi dz$	$\vec{\nabla}\phi = \hat{r}\frac{\partial\phi}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{k}\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}$ $\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(A_\varphi)$ $\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ A_r & rA_\theta & r\sin\theta\, A_\varphi\end{vmatrix}$ $\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial\phi}{\partial\varphi}\right)$ **Elementary area** $r^2\sin\theta d\theta d\varphi$ **Elementary volume** $r^2\sin\theta\, dr d\theta d\varphi$	$h_1 \quad h_2 \quad h_3$ $1 \quad 1 \quad 1$ $1 \quad \rho \quad 1$ $1 \quad r \quad r\sin\theta$ $\vec{\nabla}\phi = \hat{e}_1\frac{1}{h_1}\frac{\partial\phi}{\partial u_1} + \cdots.$ $\vec{\nabla}.\vec{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3A_1) + \cdots.\right\}$ $\nabla^2\varphi = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \cdots.\right\}$

12. Integration

1. $\int \ln x\ dx = x(\ln x - 1) + c$
2. $\int x\, e^{ax}\, dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + c$
3. $\int x\ \ln x\ dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$
4. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
5. $\int \frac{1}{x^2-a^2}dx = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$ for $x^2 > a^2$
6. $\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\frac{a+x}{a-x} + c$ for $x^2 < a^2$

7. $\int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} \ln(x^2 \pm a^2) + c$

8. $\int \frac{x}{(x^2 \pm a^2)^n} dx = \frac{-1}{2(n-1)} \frac{1}{(x^2 \pm a^2)^{n-1}} + c$ for $n \neq 1$

9. $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln(x + \sqrt{x^2 \pm a^2}) + c$

10. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}(\frac{x}{a}) + c$

11. $\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + c$

12. $\int \sqrt{a^2 - x^2} dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}(\frac{x}{a})\right] + c$

13. $\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

14. $\int_{-\infty}^{\infty} e^{(-\frac{x^2}{2\sigma^2})} dx = \sigma\sqrt{2\pi}$

15. $\int_{-\infty}^{\infty} x^n e^{(-\frac{x^2}{2\sigma^2})} dx = \begin{cases} 1 \times 3 \times 5 \times \ldots (n-1)\sigma^{n+1}\sqrt{2\pi} & for\ \boldsymbol{n} \geq \boldsymbol{2}\ and\ \boldsymbol{even} \\ 0 & for\ \boldsymbol{n} \geq \boldsymbol{1}\ and\ \boldsymbol{odd} \end{cases}$

16. $\int \sin mx \ \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \qquad m^2 \neq n^2$

17. $\int \cos mx \ \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c \qquad m^2 \neq n^2$

18. $\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^n \theta \, d\theta = \frac{n-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^{n-2} \theta \, d\theta$

Hence it can be reduced to one of the following $\int_0^{\frac{\pi}{2}} sin\,\theta\, cos\,\theta \, d\theta = \frac{1}{2}, \int_0^{\frac{\pi}{2}} sin\,\theta \, d\theta = 1, \int_0^{\frac{\pi}{2}} cos\,\theta \, d\theta = 1, \int_0^{\frac{\pi}{2}} d\theta = \boldsymbol{\frac{\pi}{2}},$

19. If $I_n = \int_0^\infty x^n e^{(-\alpha x^2)} dx$ then $I_n = \frac{n-1}{2\alpha} I_{n-2}$ with $I_0 = \frac{1}{2}\sqrt{\boldsymbol{\frac{\pi}{\alpha}}}, I_1 = \frac{1}{2\alpha}$

20. $\int f(x)g(x)dx = f(x)\int \boldsymbol{g(x)}dx - f'(x)\int(\int g(x)dx)\, dx + f'^{(x)\int\{\int(\int g(x)dx)\, dx\}}dx - \cdots ..$

13. Legendre's equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + l(l+1)y = 0$$

Solution of which is Legendre's polynomial $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$

$P_0(x) = 1,\ \ P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1)\ldots..$

a) *Coulombs law and principle of superposition leading to the definition of electrostatic field. Field* lines

The fundamental problem electrodynamics hopes to solve is this (Fig. 1): We have some electric charges (call them source charges); what force do they exert on another charge, Q (call it the test charge)? The positions of the source charges are given (as functions of time); the trajectory of the test particle is to be calculated. In general, both the source charges and the test charge are in motion.

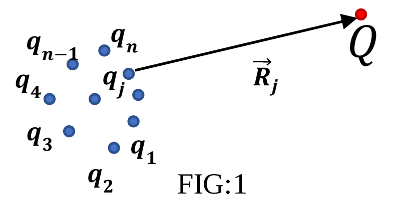


FIG:1

Well, at first sight this looks very easy: Why don't I just write down the formula for the force on Q due to q, and be done with it? But you would be shocked to see it that, for not only does the force on Q depend on the separation distance $|\vec{R}_j|$ between the charges, it also depends on both their velocities and on the acceleration of q_j. Moreover, it is not the position, velocity, and acceleration of q_j right now that matter: electromagnetic "news" travels at the speed of light, so what concerns Q is the position, velocity, and acceleration q had at some earlier time, when the message left. Therefore, in spite of the fact that the basic question ("What is the force on Q due to q_j?") is easy to state, it does not pay to confront it head on; rather, we shall go at it by stages. In the meantime, the theory we develop will allow for the solution of more subtle electromagnetic problems that do not present themselves in quite this simple format. **To begin with, we shall consider the special case of electrostatics in which all the source charges are stationary (though the test charge may be moving).**

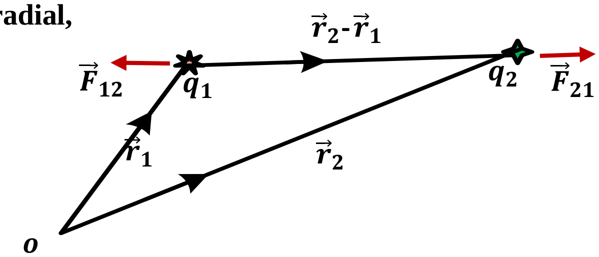
Coulomb's law:

Between 1785 and 1787, the French physicist Charles Augustine de Coulomb performed a series of experiments involving electric charges, and eventually established what is nowadays known as *Coulomb's law*.

According to this law, the force acting between two electric charges is **radial, inverse-square, and proportional to the product of the charges**.

Two like charges repel one another, whereas two unlike charges attract.

Suppose that two charges, q_1 and q_2, are located at position vectors $\vec{r_1}$ and $\vec{r_2}$ **in free space**. The electrical force acting on the second charge due to first charge is written $\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\vec{r_2}-\vec{r_1}}{|\vec{r_2}-\vec{r_1}|^3}$ in vector notation.



An equal and opposite force $\vec{F}_{21}$ acts on the first charge, in accordance with Newton's third law of motion. The SI unit of electric charge is the coulomb (C). The magnitude of the charge on an electron is 1.6022×10^{-19} C. The universal constant ϵ_0 is called the ***permittivity of free space***, and takes the value $\epsilon_0 = 8.8542\times 10^{-12}$ $C^2 N^{-1} m^{-2}$

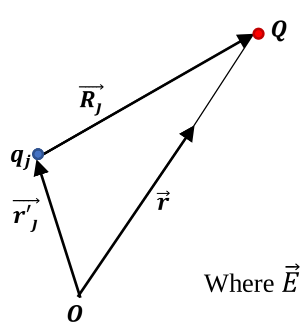
Principle of Superposition

The total force on a charge due to a number of other charges is given by the vector sum of the Coulomb forces exerted on the charge due to each of other charges acting separately. This fact is called principle of superposition. Thus if we have N point charges are $q_1, q_2, q_3, \dots \dots \dots q_n$. The forces acting on a charge Q due to all these N point charges can be found out separately $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots \dots \dots \vec{F}_n$ on Q respectively, let the position of Q and q_j's are $\vec{r}$ and $\overrightarrow{r'_j}$

So total force on due q_0 due to N charges would be $\overrightarrow{F_0} = \vec{F}_{01} + \vec{F}_{02} + \vec{F}_{03} + \cdots \dots + \vec{F}_{0n} = \sum_{j=1}^{n} \vec{F}_{0j}$

$$= \frac{Qq_1}{4\pi\epsilon_0}\frac{\vec{r}-\overrightarrow{r\prime_1}}{|\vec{r}-\overrightarrow{r\prime_1}|^3} + \frac{Qq_2}{4\pi\epsilon_0}\frac{\vec{r}-\overrightarrow{r\prime_2}}{|\vec{r}-\overrightarrow{r\prime_2}|^3} + \cdots \dots + \frac{Qq_n}{4\pi\epsilon_0}\frac{\vec{r}-\overrightarrow{r\prime_n}}{|\vec{r}-\overrightarrow{r\prime_n}|^3}$$

$$= \frac{Q}{4\pi\epsilon_0}\sum_{j=1}^{n} q_j \frac{\vec{r}-\overrightarrow{r\prime_j}}{\left|\vec{r}-\overrightarrow{r\prime_j}\right|^3} = \frac{Q}{4\pi\epsilon_0}\sum_{j=1}^{n} q_j \frac{\hat{R}_j}{\left|\vec{R}_j\right|^2} = Q\vec{E}$$



Where $\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n} q_j \frac{\hat{R}_j}{|\vec{R}_j|^2}$, is the electric field at $\vec{r}$ where Q is situated. Here $\vec{r} = \overrightarrow{r'_j} + \vec{R}_j$

$\vec{\boldsymbol{E}}$ is called the **electric field** of the source charges. Notice that it is a function of position ($\vec{\boldsymbol{r}}$), because the separation vectors $\vec{R}_j$ depend on the location of the **field point** P. But it makes no reference to the test charge Q. The electric field is a vector quantity that varies from point to point and is determined by the configuration of source charges; physically, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ is the force per unit charge that would be exerted on a test charge, if you were to place one at P.

❖ We may define electric field intensity at a point as the **limiting** force per unit charge placed at that point.

i.e. $\bar{E} = \lim\limits_{q\to 0} \frac{\bar{F}}{q}$

Note that q→0 is an idealization because it contradicts the quantization of charge. In case of immobile charges, we need not require this idealization.

Coulomb's law and the principle of superposition constitute the physical input for electrostatics—the rest, except for some special properties of matter, is mathematical elaboration of these fundamental rules.

Continuous charge distribution.

If Δq is the amount of charge in a small volume ΔV about any point at $\vec{r'}$ the volume charge density $\boxed{\boldsymbol{\rho} = \lim\limits_{\Delta V\to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV}}$

If Δq is the amount of charge in a small surface ΔS about any point at $\vec{r'}$ the surface charge density $\boxed{\boldsymbol{\sigma} = \lim\limits_{\Delta S\to 0} \frac{\Delta q}{\Delta S} = \frac{dq}{dS}}$

If Δq is the amount of charge in a small line Δl about any point at $\vec{r'}$ the line charge density $\boxed{\boldsymbol{\lambda} = \lim\limits_{\Delta l\to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}}$

A point charge can be described with a charge density as $\boxed{\boldsymbol{\rho}(\overline{\boldsymbol{r'}}) = \boldsymbol{q_1}\boldsymbol{\delta}(\overline{\boldsymbol{r'}} - \overline{\boldsymbol{r_1}})}$

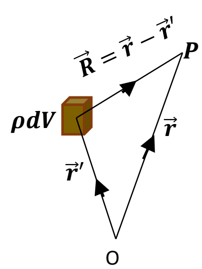
(**Where $\delta(\overline{r'} - \overline{r_1})$ is a Dirac delta function. And here the point charge q_1 is situated at the position $\overline{r_1}$**)

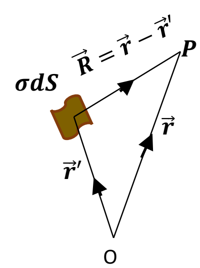
Electric field due to charge distribution

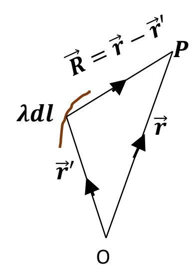
For volume charge distribution $\vec{E} = \frac{1}{4\pi\epsilon_0} \iiint \rho dV \frac{\vec{R}}{|\vec{R}|^3}$

For surface charge distribution $\vec{E} = \frac{1}{4\pi\epsilon_0} \iiint \sigma dS \frac{\vec{R}}{|\vec{R}|^3}$

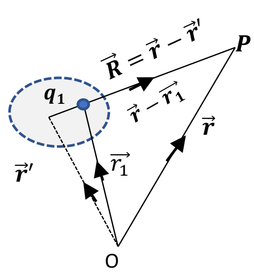
For linear charge distribution $\vec{E} = \frac{1}{4\pi\epsilon_0} \iiint \lambda dl \frac{\vec{R}}{|\vec{R}|^3}$







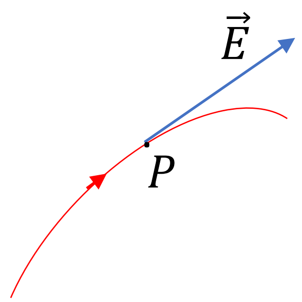
For point charge distribution $\vec{E} = \frac{1}{4\pi\epsilon_0} \iiint \rho dV \frac{\vec{R}}{|\vec{R}|^3} = \frac{1}{4\pi\epsilon_0} \iiint \boldsymbol{q_1\delta}(\overline{\boldsymbol{r'}} - \overline{\boldsymbol{r_1}}) dV \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{\boldsymbol{q_1}}{4\pi\epsilon_0} \frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r_1}}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r_1}}|^3}$

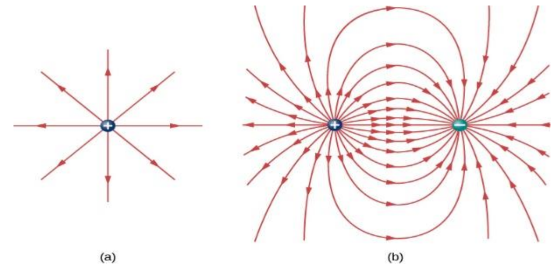


FIELD LINES

Electric field lines are a pictorial description of the electric field, developed by the great 19th century physicist Michael Faraday. An electric line of force is an imaginary curve drawn in such a way that the **tangent to this curve at any point gives the direction of electric field at the point.** The strength (magnitude) of the electric field is indicated by **the number of electric field lines** in a region - lots of closely spaced lines means a strong electric field, a few widely spaced lines mean a weak electric field. The magnitude of the electric field is proportional to the number of electric field lines.

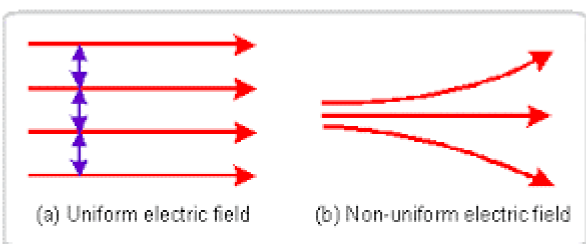
Since the number of electric field lines is proportional to the strength of the electric field, if the electric field has the same strength at all points in a region the electric field lines will be parallel in that region.

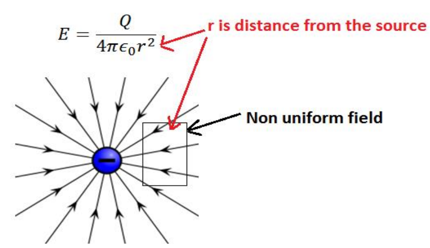




The electric field line diagram of a positive point charge

The field line diagram of a dipole.





What is electric line of forces of field lines?

An electric line of force is the imaginary curve drawn in such a way that the tangent of this curve at any point gives the direction of electric field at that point.

Differential equation of lines of force.

Let $\vec{r} = \vec{r}(s)$ represent a space curve. $\frac{d\vec{r}}{ds}$ is a vector tangent to this curve at every point. If this curve represent a line of force, $\frac{d\vec{r}}{ds}$ must be parallel to the local electric field $\vec{E}$, i.e., $\frac{d\vec{r}}{ds} = \alpha\vec{E}$ where α is a constant.

In component form

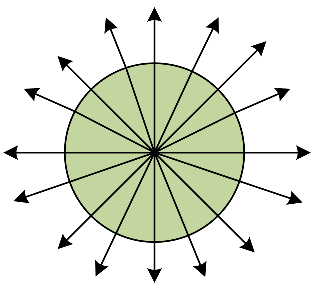
$$\frac{dx}{ds} = \alpha E_x \; ; \; \frac{dy}{ds} = \alpha E_y \; ; \; \frac{dz}{ds} = \alpha E_z$$

$$\therefore \; \frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$$

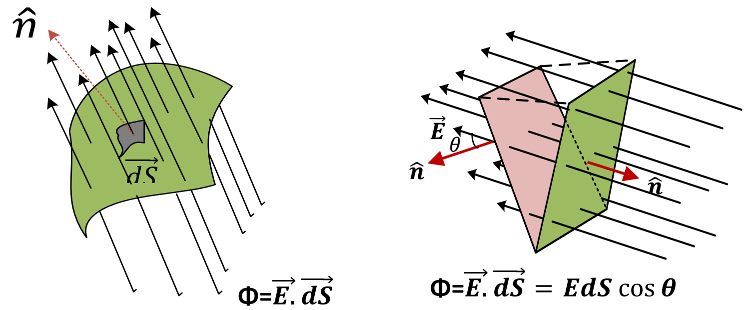
$\frac{dy}{dx} = \frac{E_Y}{E_X}$ in 2D Cartesian co-ordinate and similarly $\frac{dr}{d\theta} = r\frac{E_r}{E_\theta}$ in plane polar co-ordinate.

b) Divergence of electrostatic field. Flux, Gausses theorem of electrostatic. Application of Gausses theorem to find the electric field due to charge configurations with spherical, cylindrical and planer symmetry.

FLUX Electric flux is the rate of flow of the electric field through a given area. **Electric flux is proportional to the number of electric field lines going through a virtual surface.** Flux $\boldsymbol{\Phi}=\vec{E}.\vec{dS}$



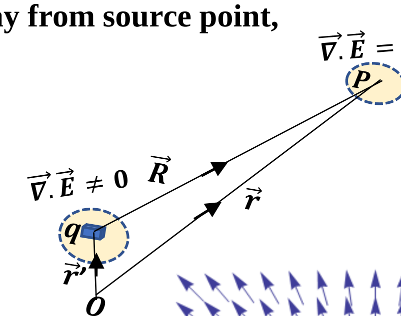
Flux through closed surface



flux through open surface

If for any charge q, we consider that **the field point (P) is far away from source point,** then electric field due to the charge can be expressed as

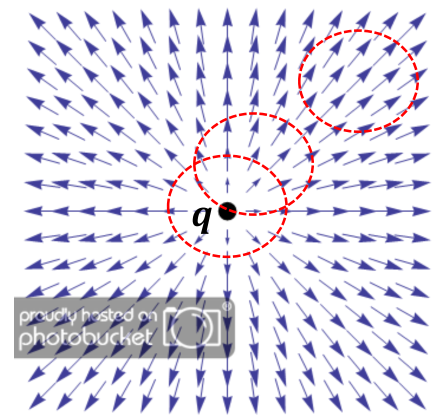
$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \quad [\text{ As then } \vec{R} = \vec{r} - \vec{r}' \approx \vec{r}\ ]$$



Then divergence of electrostatic field is

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} = \frac{q}{4\pi\epsilon_0}\ \vec{\nabla}.\frac{\vec{r}}{r^3} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^3}\vec{\nabla}.\vec{r} + \vec{r}.\vec{\nabla}\left(\frac{1}{r^3}\right)\right]$$
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{3}{r^3} + \vec{r}.(-3r^5\vec{r})\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{3}{r^3} - \frac{3}{r^3}\right] = 0$$

$\boxed{\vec{\nabla}.\vec{E} = 0}$ **at a point where there is no charge**

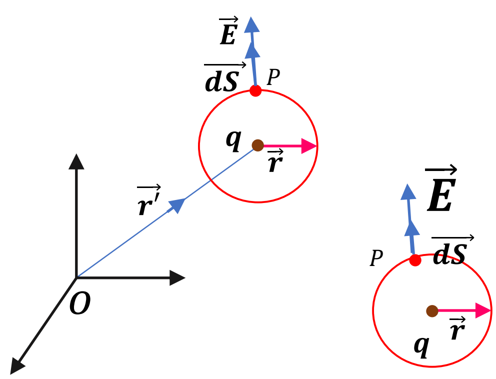


Hence divergence of electric field **far away from the source is 'zero'.** According to definition of divergence of any vector field that mean the outward flux per unit volume from an infinitesimal volume around point P is zero. Now if we consider an infinitesimal volume whose boundary surface enclosing the source point then **we get different result.**

Hence what about a volume that enclose a charge? Suppose I have charge q at $\vec{r}'$ and I make a spherical volume of radius r which is easiest to handle around the charge.

Then as $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$ at surface then $\vec{E}.\vec{dS}=EdS$

Where $\vec{dS}$= is an elementary surface on spherical surface.



Then $\oint\!\!\oint \vec{E}.\vec{dS} = \oint\!\!\oint EdS = \oint\!\!\oint \frac{q}{4\pi\epsilon_0}\frac{r^2\sin\theta\, d\theta\, d\varphi}{r^2} = \frac{q}{4\pi\epsilon_0}\times 4\pi = \frac{q}{\epsilon_0}$

$\oint\!\!\oint \vec{E}.\vec{ds} = \frac{q}{\epsilon_0}$ **this is known as Gausses law**

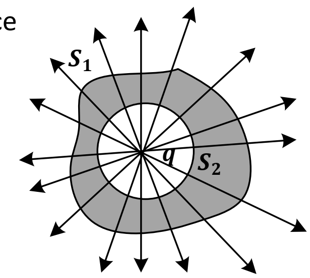
$\oint\!\!\oint \vec{E}.\vec{ds} = \frac{1}{\epsilon_0}q_{in} = \iiint \frac{\rho}{\epsilon_0}\, dv$, where ρ is the volume charge density

Now applying divergence theorem we have $\iiint \boldsymbol{\nabla}.\vec{E}\, dv = \iiint \frac{\rho}{\epsilon_0}\, dv$……………………………….(I)

As equation (i) holds good for any arbitrary volume dv then $\boxed{\boldsymbol{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}}$ **(this is differential form of Gauss's law)**

Now we take here a spherical surface what will happen for any surface of arbitrary shape

Let S_1 is any surface of arbitrary shape. It enclose a charge q. Now consider a spherical surface S_2 enclosing q inside the surface S_1. The flux through S_2 is equal to flux through S_1 as the shaded region will contribute nothing to the flux as there is no charge inside the shaded region. Hence for any arbitrary surface $\oiint \bar{E}.\overline{ds} = \frac{q}{\epsilon_0}$ is also true. Hence we will get also

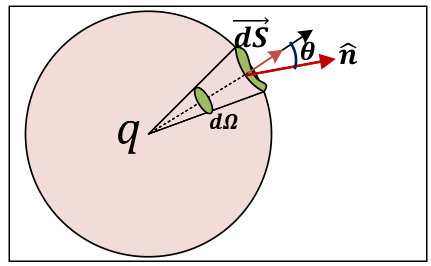


$\nabla.\bar{E} = \frac{\rho}{\epsilon_0}$

State and prove Gausse's law in electrostatic field.

In an arbitrary electrostatic field (in vacuum) the total; electric flux over any close surface is equal to $1/\epsilon_0$ times the total charge (q_{in}) enclosed by the surface, where ϵ_0 is the free space permittivity

Mathematically $\oiint \bar{E}.\overline{ds} = \frac{1}{\epsilon_0} q_{in}$



Suppose a point charge q located at the point O and draw a close surface. Let $\overline{ds}$ be any elementary surface at position vector $\bar{r}$ of any point P w.r.t O.

Let the electric field at P be $\bar{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$

The flux of $\bar{E}$ through $\overline{ds}$ is $\bar{E}.\overline{ds}$.

Then total flux $\oiint \bar{E}.\overline{ds} = \oiint \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}.(ds\,\hat{n})$

$$= \frac{Q}{4\pi\epsilon_0}\oiint ds\hat{n}.\frac{\hat{r}}{r^2} = \frac{Q}{4\pi\epsilon_0}\oiint \frac{ds\cos\theta}{r^2} = \frac{Q}{4\pi\epsilon_0}\oiint d\Omega$$

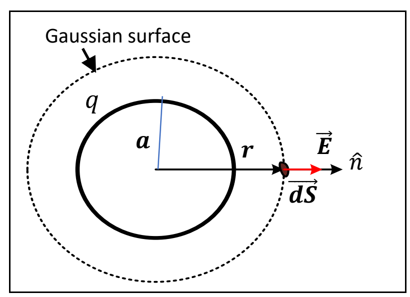
$$= \frac{Q}{4\pi\epsilon_0} 4\pi = \frac{Q}{\epsilon_0}$$

Hence $\oiint \bar{E}.\overline{ds} = \frac{1}{\epsilon_0} q_{in}$.

Application of Guess's Theorem:

Calculation of Electric Field

a) **Uniformly charged hollow sphere**: Consider a uniformly charged hollow sphere of radius a carrying a total charge Q. We are to find electric field at a point P at a distance r form the centre of sphere.



(i) **When $r > a$ (*P is an external point*):** To find the electric field at P we construct a concentric sphere of radius r passing through the point P as Gaussian surface. Consider an elementary area $d\vec{s}$ around P & Let electric field at P is $\vec{E}$. According to Gauss's Law,

$\oiint \vec{E}.d\vec{s} = \frac{Q_{in}}{\epsilon_0}$

[Where Q_{in} =charge enclosed by Gaussian surface, $\epsilon_0$free space permittivity]

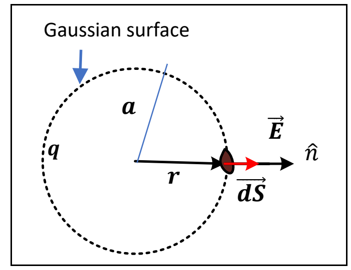
$or, \oiint \vec{E}.d\vec{s} = \frac{Q}{\epsilon_0}$ [Here $Q_{in} = Q$]

$or, \oiint Eds = \frac{Q}{\epsilon_0}$ [Here $\vec{E}$ & $d\vec{s}$ are in same direction (radially outwards)]

$or, E\oiint ds = \frac{Q}{\epsilon_0}$ [The magnitude of electric field is same at any point on Gaussian surface]

$or, E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$ Hence $\boxed{\boldsymbol{E = \frac{Q}{4\pi\epsilon_0 r^2}}}$ **which is directed radially outwards.**

(ii) **<u>When $r = a$ (P is in the surface of charged surface):</u>** To find the electric field at P we construct a concentric sphere of radius a passing through the point P as Gaussian surface. Consider an elementary surface $d\vec{s}$ around P and let electric field at P is $\vec{E}$, then accordingly to Gauss's law,



$\oint\!\!\oint \vec{E}.d\vec{s} = \frac{Q_{in}}{\epsilon_0}$ [Where Q_{in} =charge enclosed by Gaussian surface, $\epsilon_0$free space permittivity]

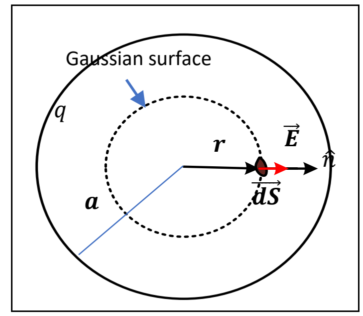
$or, \oint\!\!\oint \vec{E}.d\vec{s} = \frac{Q}{\epsilon_0}$ [Here $Q_{in} = Q$]

$or, \oint\!\!\oint Eds = \frac{Q}{\epsilon_0}$ [Here $\vec{E}$ & $d\vec{s}$ are in same direction (radially outwards)]

$or, E \oint\!\!\oint ds = \frac{Q}{\epsilon_0}$ [The magnitude of electric field is same at any point on Gaussian surface]

$or, E \cdot 4\pi a^2 = \frac{Q}{\epsilon_0}$ Hence $\boxed{\boldsymbol{E = \frac{Q}{4\pi\epsilon_0 a^2}}}$ **which is directed radially outwards.**

(iii) **<u>When $r < a$ (P is an internal point):</u>** To find the electric field at P we construct a concentric sphere of radius r passing through the point P as Gaussian surface. Consider an elementary area $d\vec{s}$ around P & Let electric field at P is $\vec{E}$. According to Gauss's Law,



$\oint\!\!\oint \vec{E}.d\vec{s} = \frac{Q_{in}}{\epsilon_0}$ [Where Q_{in} =charge enclosed by Gaussian surface, $\epsilon_0$free space permittivity]

$or, \oint\!\!\oint \vec{E}.d\vec{s} = 0$ [Here $Q_{in} = 0$]

The above equation is valid for any arbitrary surface $d\vec{s}$. Hence $\boxed{\boldsymbol{\vec{E} = \vec{0}.}}$

b) <u>Uniformly charged solid sphere:</u>

(i) **<u>When $r > a$ (P is an external point):</u>** Same as uniformly charged hollow sphere. $\boxed{\boldsymbol{E = \frac{Q}{4\pi\epsilon_0 r^2}.}}$

(ii) **<u>When $r = a$ (P is in the surface of charged surface):</u>** Same as uniformly charged hollow sphere. $\boxed{\boldsymbol{E = \frac{Q}{4\pi\epsilon_0 a^2}.}}$

(iii) **<u>When $r < a$ (P is an internal point):</u>** To find the electric field at P we construct a concentric sphere of radius r passing through the point P as Gaussian surface. Consider an elementary area $d\vec{s}$ around P & Let electric field at P is $\vec{E}$.

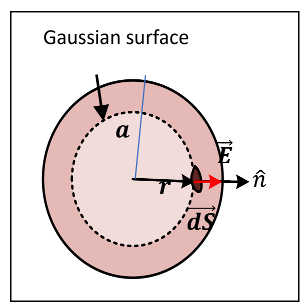
According to Gauss's Law,

$\oint\!\!\oint \vec{E}.d\vec{s} = \frac{Q_{in}}{\epsilon_0}$ [Where Q_{in} =charge enclosed by Gaussian surface, $\epsilon_0$free space permittivity]…………(i)

Here $Q = \frac{4}{3}\pi a^3 \rho$, [Where $\rho = volume\ charged\ density$]

Then $Q_{in} = \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi \frac{a^3}{a^3} r^3 \rho = \frac{Q}{a^3} r^3$.

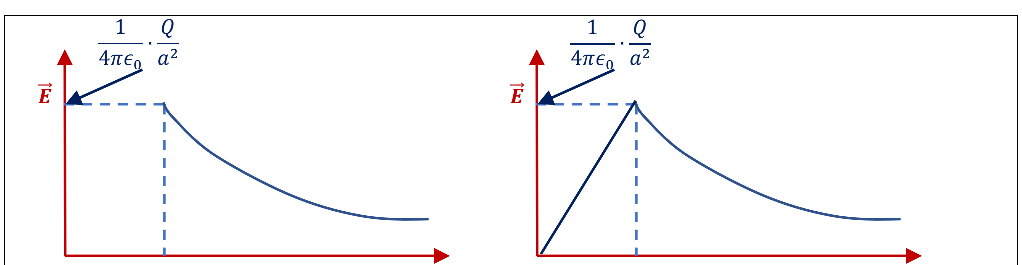
Hence from eqn (i), $\oint\!\!\oint \vec{E}.d\vec{s} = \frac{Q}{\epsilon_0 a^3} r^3$



$or, \oint\!\!\oint Eds = \frac{Q}{\epsilon_0 a^3} r^3$ [Here $\vec{E}$ & $d\vec{s}$ are in same direction (radially outwards)]

$or, E \oint\!\!\oint ds = \frac{Q}{\epsilon_0 a^3} r^3$ [The magnitude of electric field is same at any point on Gaussian surface]

$or, E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0 a^3} r^3$ Hence $\boxed{\boldsymbol{E = \frac{Q}{4\pi\epsilon_0 a^3} r}}$ [The direction same] $\Rightarrow$ $\boxed{\boldsymbol{E \propto r}}$



For uniformly charged hollow sphere

For uniformly charged solid sphere

(c) Uniformly charged hollow cylinder: Consider a long hollow cylinder of radius a carrying a uniform charge density of λ per unit length. We are to find electric field at a point P at distance r from the axis of cylinder.

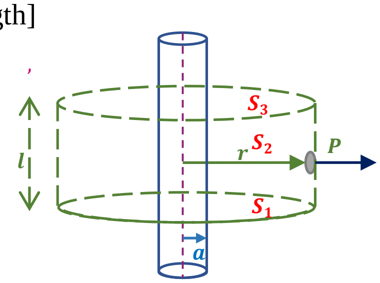
(i) **When $r > a$ (P is an external point):** To find the electric field at P we construct a concentric Gaussian surface. Which in this case is a coaxial cylinder of radius r and length l. Consider elementary area $d\vec{s}$ around P & let electric field at P is $\vec{E}$. According to Gauss's Law,

$\oiint \vec{E}.d\vec{s} = \frac{Q_{in}}{\epsilon_0}$ [Where Q_{in} =charge enclosed by Gaussian surface, $\epsilon_0$free space permittivity]

$or, \iint_{S_1} \vec{E}.d\vec{s} + \iint_{S_2} \vec{E}.d\vec{s} + \iint_{S_3} \vec{E}.d\vec{s} = \frac{\lambda l}{\epsilon_0}$

[Total surface is made by S_1, S_2, S_3 and $Q_{in} = \lambda l$, λ=Charge per unit length]

$or, 0 + \iint_{S_2} Eds + 0 = \frac{\lambda l}{\epsilon_0}$ [For S_1& S_3 surface $\vec{E} \perp d\vec{s}$ and for S_2 surface $\vec{E} \| d\vec{s}$]

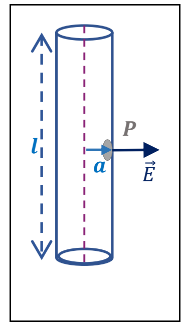


$or, E \iint ds = \frac{\lambda l}{\epsilon_0}$ [For S_2 surface E is same every point]

$or, E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0}$

$or, E = \frac{\lambda}{2\pi\epsilon_0 r}$. Hence $\boxed{\boldsymbol{E = \frac{2\lambda}{4\pi\epsilon_0 r}}}$.

(ii) **When $r = a$ (P is in the surface of charged cylinder):** To find the electric field at P we construct a concentric cylinder of radius a passing through the point P as Gaussian surface and consider a elementary surface at P and let electric field at P is $\vec{E}$, then according to Gauss's law,



$\oiint \vec{E}.d\vec{s} = \frac{\lambda l}{\epsilon_0}$ [λ=Charge per unit length]

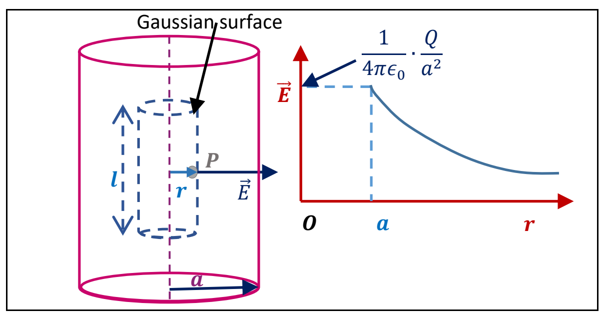
$or, \oiint Eds = \frac{\lambda l}{\epsilon_0}$ [Here $\vec{E}$ & $d\vec{s}$ are in same direction (radially outwards)]

$or, E \oiint ds = \frac{\lambda l}{\epsilon_0}$ [The magnitude of electric field is same at any point on Gaussian surface]

$or, E \cdot 2\pi a l = \frac{\lambda l}{\epsilon_0}$

$or, E = \frac{\lambda}{2\pi\epsilon_0 a} \Rightarrow E = \frac{2\lambda}{4\pi\epsilon_0 a}$ In vector form, $\boxed{\boldsymbol{\vec{E} = \frac{2\lambda}{4\pi\epsilon_0 a}\hat{r}}}$ Which is directed radially outwards.

(iii) **When $r < a$ (P is an internal point):** To find the electric field at P we construct a concentric cylinder of radius r passing through the point P as Gaussian surface. Consider an elementary area $d\vec{s}$ around P & Let electric field at P is $\vec{E}$. According to Gauss's Law,



$\oiint \vec{E}.d\vec{s} = \frac{Q_{in}}{\epsilon_0}$ [Where Q_{enc} =charge enclosed by Gaussian surface, $\epsilon_0$free space permittivity]

$or, \oiint \vec{E}.d\vec{s} = 0$ [Here $Q_{enc} = 0$]

The above equation is valid for any arbitrary surface $d\vec{s}$.

Hence $\boxed{\boldsymbol{\vec{E} = \vec{0}}}$.

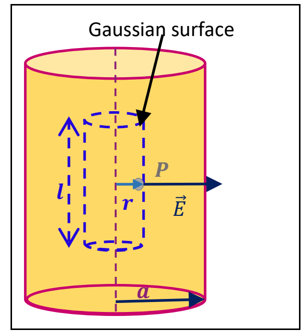
(d) Uniformly charged solid cylinder:

(i) **When $r > a$ (P is an external point)**: Same as uniformly charged hollow cylinder. $\boxed{\boldsymbol{E} = \frac{2\lambda}{4\pi\epsilon_0 r}.}$

(ii) **When $r = a$ (P is in the surface of charged surface):**

Same as uniformly charged hollow sphere. $\boxed{\vec{\boldsymbol{E}} = \frac{2\lambda}{4\pi\epsilon_0 a}\hat{\boldsymbol{r}}.}$

(iii) **When $r < a$ (P is an internal point):** To find the electric field at P we construct a concentric cylinder of radius r passing through the point P as Gaussian surface. Consider an elementary area $d\vec{s}$ around P & Let electric field at P is $\vec{E}$.



According to Gauss's Law,

$\oiint \vec{E}.d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$

[Where Q_{enc} =charge enclosed by Gaussian surface, ϵ_0 free space permittivity]…….(i)

Here $Q = \pi a^2 l\rho$, [Where $\rho = volume\ charged\ density$]

Then $Q_{enc} = \pi a^2 l\rho = \pi\frac{a^2}{a^2}r^2 l\rho = \frac{Q}{a^2}r^2$. Hence from eqn(i),

$\oiint \vec{E}.d\vec{s} = \frac{Q}{\epsilon_0 a^2}r^2$

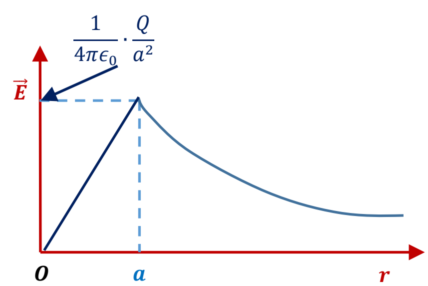
$or, \oiint Eds = \frac{Q}{\epsilon_0 a^2}r^2$ [Here $\vec{E}$ & $d\vec{s}$ are in same direction (radially outwards)]

$or, E\oiint ds = \frac{Q}{\epsilon_0 a^2}r^2$ [The magnitude of electric field is same at any point on Gaussian surface]

$or, E\cdot 2\pi rl = \frac{Q}{\epsilon_0 a^2}r^2$

$or, E = \frac{Qr}{2\pi\epsilon_0 la^2} = \frac{\lambda lr}{2\pi\epsilon_0 la^2}$ [Where λ=Charge per unit length]

$or, E = \frac{\lambda}{2\pi\epsilon_0 a^2}r = \frac{2\lambda}{4\pi\epsilon_0 a^2}r$ In vector form, $\boxed{\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\cdot\frac{2\lambda r}{a^2}\hat{\boldsymbol{r}}}$

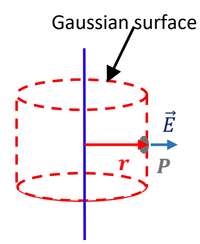


For uniformly charged solid cylinder

(e) Uniformly charged wire:

When $r > a$ (P is an external point):

Same as uniformly charged hollow/solid cylinder. $\boxed{\boldsymbol{E} = \frac{2\lambda}{4\pi\epsilon_0 r}.}$



(f) Uniformly charged infinite plane sheet of charge:

Let us consider a charged infinite plane having a surface density σ (considering both side). Suppose we are to find field at any point P, we construct the Gaussian surface, which in this case is a cylinder whose base area is A, whose axis is perpendicular to the surface and extending equal distance on the two sides of the plane. From Gauss's law,

$\oiint \vec{E}.d\vec{s} = \frac{Q_{in}}{\epsilon_0}$ [Where Q_{in} =charge enclosed by Gaussian surface, $\epsilon_0$free space permittivity]

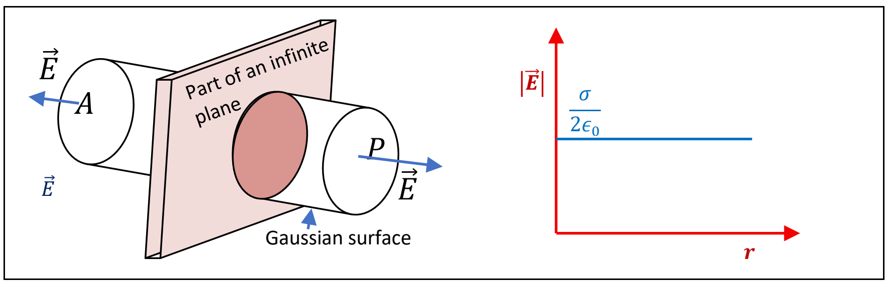
$or, \iint_{S_1} \vec{E}.d\vec{s} + \iint_{S_2} \vec{E}.d\vec{s} + \iint_{S_3} \vec{E}.d\vec{s} = \frac{\sigma A}{\epsilon_0}$

[Total surface is made by S_1, S_2, S_3 and $Q_{in} = \sigma A$, σ=Charge per unit surface]

$or, \iint_{S_1} Eds + 0 + \iint_{S_3} Eds = \frac{\sigma A}{\epsilon_0}$ [For S_2 surface $\vec{E} \perp d\vec{s}$ and for S_1& S_3 surface $\vec{E} \| d\vec{s}$]

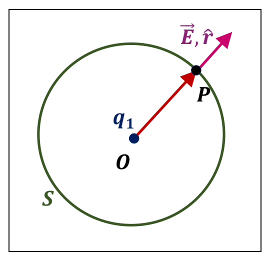
$or, E \iint ds + E \iint ds = \frac{\sigma A}{\epsilon_0}$ [For S_1& S_3 surface E is same every point]

$or, EA + EA = \frac{\sigma A}{\epsilon_0}$ Hence $\boxed{\boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{2\epsilon_0}}}$. Which is directed perpendicular to the charged surface. E is constant.



Coulomb's law from Gauss's law:

We consider a point charge q_1 placed at O and construct a spherical Gaussian surface (S) of radius r with its centre at O. For this surface Gauss's law states that, $\boxed{\int_S \vec{E}.d\vec{S} = \frac{1}{\epsilon_0} \cdot q_1}$. Because of spherical symmetry electric field lines will be normal to the Gaussian surface and $\vec{E}$ will have constant magnitude all over the Gaussian surface.



$\therefore \int_S \vec{E}.d\vec{S} = E \int dS = E \cdot 4\pi r^2$

Now, $E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \cdot q_1 \Rightarrow E = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2}$

In vector form, $\vec{\boldsymbol{E}} = \frac{\boldsymbol{1}}{\boldsymbol{4\pi\epsilon_0}} \cdot \frac{\boldsymbol{q_1}}{\boldsymbol{r^2}} \hat{\boldsymbol{r}}$ **The electric field at any point P with position vector $\vec{r}$ with respect to the position of q_1 .**

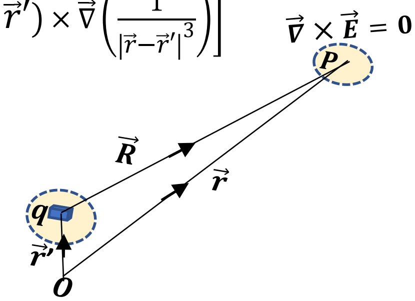
If a second point charge q_2 is placed at P then the force on it due to q_1 will be, $\vec{F} = q_2\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r^2} \hat{r}$

This is the Coulomb's law of force between two point charges.

c) Curl of the electrostatic field and its conservative nature. Electric potential. Potential for a uniformly charged spherical shell and solid sphere. Calculation of electric field from potential.

We first consider the electric field $\vec{E}$ at the position $\vec{r}$ due to a point charge q located at $\vec{r}'$, is $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$

Now $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{q}{4\pi\epsilon_0}\vec{\nabla}\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{q}{4\pi\epsilon_0}\left[\frac{\vec{\nabla}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + (\vec{r}-\vec{r}')\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|^3}\right)\right]$



Now using Cartesian co-ordinate it can be shown by direct calculation that

$\vec{\nabla}\times(\vec{r}-\vec{r}') = \vec{0}$ & $\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|^3}\right) = -\frac{3(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

Substituting this result, we get $\vec{\nabla}\times\vec{E} = \vec{0}$

From the principle of superposition, we know that the total field due to any charge distribution is given by the vector sum of the field's due to individual charges. So we can say that the result holds for any charge distribution. Therefore, we can write $\vec{E} = -\vec{\nabla}\phi$..............................(i)

since, $\vec{\nabla}(\phi + constant) = \vec{\nabla}\phi$, the potential defined by eqn(i) is arbitrary by some additive constant. Addition of a constant to yields the same electric field. Moreover, it does not affect the potential difference between two points because the constants cancel out. The absolute value of potential is of no importance; only potential difference has physical significance. To get a physical significance of the electrostatic potential let us calculate the work done **against** the field in moving a unit positive charge from a reference point a to the point b.

This is $W_{ab} = -\int_a^b \vec{E}.\vec{dr} = \int_a^b \vec{\nabla}\phi.\vec{dr} = \int_a^b d\phi = \phi(b) - \phi(a)$............(ii)

Obviously the work done depends on the on the positions of a and b and not on the path connecting them. This indicates that the **work done over any closed but is zero. This result can be easily obtained from by using strokes theorem.** $\oint_C \vec{E}.\vec{dr} = \iint_S(\vec{\nabla}\times\vec{E}).\vec{dr} = \mathbf{0}$ **[Where C is the contour bounding an open surface S.]**

This must be so because **electrostatic force is conservative.** Equation (ii) suggests that the electrostatic potential maybe considered as potential energy per unit charge.

If the charge distribution that creates the electric field is localized in a finite region of space that electric field vanishes at in finite distance from the charge distribution within one usually takes the reference point a at infinity, where the potential is taken to be zero $\phi(\infty) = 0$, in this case equation (ii) gives $\phi(b) = -\int_\infty^b \vec{E}.\vec{dr}$

Now, as the electric field is the force per unit positive charge, the electrostatic potential at any point(b) maybe defined **as the work done by an external agency in bringing a unit positive charge from infinity to that point**

If the same reference point is chosen for the electrostatic potential and potential energy, **then potential energy of a charge is given by the product of the charge and the electric potential at the location of the charge.**

Potential due to a point charge

the electric field $\overrightarrow{E}$ at the position $\vec{r}$ due to a point charge q located at $\vec{r}'$, is $\overrightarrow{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$

now from vector analysis it can be shown that $\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

Therefore equation can be rewritten as $\overrightarrow{E} = -\frac{q}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\vec{\nabla}\left(\frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}\right)$

Comparing it with the relation $\overrightarrow{E} = -\vec{\nabla}\,\phi$ we get the electrostatic potential due to the point charge as $\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{\vec{r}-\vec{r}'}$

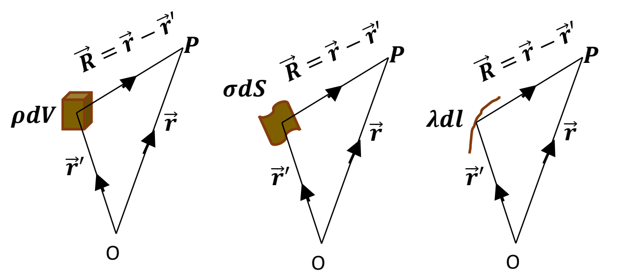
Potential due to charge distribution just like electrostatic force on a test charge and electrostatic field the electrostatic potential also **obeys the principle of superposition.**

So using this principle it is possible to calculate potential due to arbitrary charge distribution.

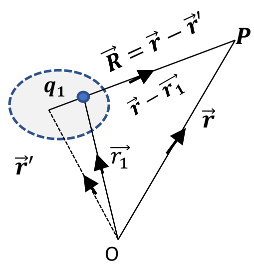
For volume charge distribution $\phi = \frac{1}{4\pi\epsilon_0}\iiint \rho dV \frac{1}{|\vec{R}|}$

For surface charge distribution $\phi = \frac{1}{4\pi\epsilon_0}\iiint \sigma dS \frac{1}{|\vec{R}|}$

For linear charge distribution $\phi = \frac{1}{4\pi\epsilon_0}\iiint \lambda dl \frac{1}{|\vec{R}|}$



For point charge distribution $\phi = \frac{1}{4\pi\epsilon_0}\iiint \rho dV \frac{1}{|\vec{R}|} = \frac{1}{4\pi\epsilon_0}\iiint \boldsymbol{q_1\delta(\overrightarrow{r'} - \overrightarrow{r_1})}dV\frac{1}{|\vec{r}-\vec{r}'|} = \frac{\boldsymbol{q_1}}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r_1}|}$



Equipotential surfaces

An equipotential surface is the locus of points having the same potential in case of a point charge located at the origin, the potential at a distance r from the origin(**far away from charge location**) is given by $\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

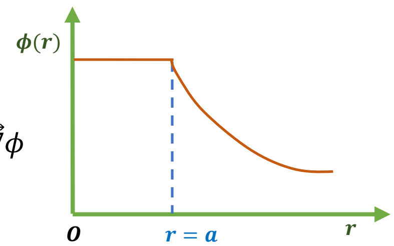
obviously a sphere of radius r with the Centre at Q will be equipotential surfaces. In general, the shape of equipotential surface depends on the charge configuration. But whatever may be its shape that relation $\overrightarrow{E} = -\vec{\nabla}\,\phi$ indicates that electric field $\overrightarrow{E}$ is always perpendicular to the equipotential surface $\phi = constant$, at every point.

Advantage of potential concept: Electric field $\overrightarrow{E}$ is a vector quantity. It's direct calculation very often because tedious and cumbersome. On the other hand, potential ϕ is a scaler quantity. In many cases it's calculation is found to be easier. The potential concept reduces a vector problem down to a scaler one. So in practice it is often preferable to determine $\overrightarrow{E}$ by first calculating ϕ and then using relation $\overrightarrow{E} = -\vec{\nabla}\,\phi$ calculate instead of determining $\overrightarrow{E}$ directly.

Units of electric field and potential: In SI unit, unit of electric field is **NEWTON/COULOMB** or, **VOLT/METER**

Where unit of potential is in SI unit **VOLT** or **JOULE/COULOMB.**

Calculation of Potential



Uniformly charged hollow sphere:

(i) **When $r \geq a$ (For an external point):** Field and potential is related as, $\vec{E} = -\vec{\nabla}\phi$

Then, $\frac{Q}{4\pi\epsilon_0 r^2}\hat{r} = -\hat{r}\frac{d\phi}{dr} \Rightarrow \int d\phi = -\int \frac{Q}{4\pi\epsilon_0 r^2}dr$

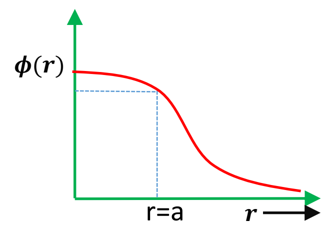
$\Rightarrow \phi(r) = -\left(-\frac{Q}{4\pi\epsilon_0 r}\right) + c \Rightarrow \phi(r) = \frac{Q}{4\pi\epsilon_0 r} + c$ [∴c=integral constant]

Now at $r = \infty$, $\phi(r)$ then $c = 0$, hence $\boldsymbol{\phi(r) = \frac{Q}{4\pi\epsilon_0 r}}$ for $r \geq a$ and $\boldsymbol{\phi(r)|_{r=a} = \frac{Q}{4\pi\epsilon_0 a}}$

(ii) When $r < a$ (For an internal point):
As fields and potential are related as, $\vec{E} = -\vec{\nabla}\phi$. Then $\frac{d\phi}{dr} = 0$, hence $\phi = const = C_1(say)$.

Now as $\phi(r)$ is continuous across the boundary $r = a$ as we have ,

$\phi(r)|_{r=a} = \frac{Q}{4\pi\epsilon_0 a}$ then $\frac{Q}{4\pi\epsilon_0 a} = C_1$. Hence at any internal point $\boldsymbol{\phi = \frac{Q}{4\pi\epsilon_0 a}}$.



Uniformly charged solid sphere:

(i) **When $r \geq a$ (For external point):** Same as uniformly charged hollow sphere.

$\boldsymbol{\phi(r) = \frac{Q}{4\pi\epsilon_0 r}}$ & $\boldsymbol{\phi(r)|_{r=a} = \frac{Q}{4\pi\epsilon_0 a}}$.

(ii) **When $r < a$ (For an internal point):**
We know the relation between electric field and potential is $\vec{E} = -\vec{\nabla}\phi$.

Then, $\frac{Q}{4\pi\epsilon_0 a^3}r\hat{r} = -\hat{r}\frac{d\phi}{dr} \Rightarrow \int d\phi = -\frac{Q}{4\pi\epsilon_0 a^3}\int r\,dr \Rightarrow \phi = -\frac{Q}{4\pi\epsilon_0 a^3}\frac{r^2}{2} + c$ [∴c=integral constant]

When $r = a$, $\phi = \frac{Q}{4\pi\epsilon_0 a}$ then $\frac{Q}{4\pi\epsilon_0 a} = -\frac{Q}{4\pi\epsilon_0 a^3}\frac{a^2}{2} + c \Rightarrow c = \frac{3}{2}\frac{Q}{4\pi\epsilon_0 a}$

Then $\phi = -\frac{Q}{4\pi\epsilon_0 a^3}\frac{r^2}{2} + \frac{3}{2}\frac{Q}{4\pi\epsilon_0 a} \Rightarrow$ $$\phi = \frac{Q}{4\pi\epsilon_0 a}\left[\frac{3}{2} - \frac{r^2}{2a^2}\right].$$

d) Laplace's and Poisson equation. Uniqueness theorems. Method of images and its application to (1) plane infinite sheet and (2) sphere.

Starting from Gauss's law establish Poisson equation in electrostatic and what is Laplace's modification?

$$\oiint \bar{E}.\overline{ds} = \frac{1}{\epsilon_0} q_{in} = \iiint \frac{\rho}{\epsilon_0}\, dV \qquad , where\ \rho\ is\ the\ volume\ charege\ density$$

Now applying divergence theorem we have $\iiint \nabla.\bar{E}\, dV = \iiint \frac{\rho}{\epsilon_0}\, dV$……………………………..(I)

As equation (i) holds good for any arbitrary volume dV then $\boldsymbol{\nabla.\bar{E}} = \boldsymbol{\frac{\rho}{\epsilon_0}}$ (this is differential form of Gauss's law)

Now as electrostatic field is conservative then, $\bar{E} = -\nabla\phi$ (where $\boldsymbol{\phi}$ is corresponding electrostatic potential.)

Then $\boldsymbol{\nabla^2\phi = -\frac{\rho}{\epsilon_0}}$ (this is Poisson's equation.)

And for free space, $\rho = 0\ then\ \boldsymbol{\nabla^2\phi = 0}$ (this is Laplace's equation.).

State and explain Uniqueness theorem.

"The solution ϕ to the Laplace Equation $\nabla^2\phi = 0$ in some volume V is uniquely determined if ϕ is specified on the boundary surface."

There is a "second uniqueness theorem" that says that in a volume V surrounded by conductors and containing a specified charge densityρ, the electric field is uniquely determined in the volume if the total charge on each conductor is given.

The uniqueness theorems for the Poisson Equation is similar to that of the Laplace Equation except that in addition the distribution of space-charge density throughout the volume ($\rho(x, y, z)$) is also required to constrain the solution.
If the volume contains linear dielectric material, then the shapes and dielectric constants of that material are also required.
The uniqueness theory is a license to my imagination. It doesn't matter how I come by my solution; if (i) it satisfies Laplace's equation (ii) it has correct values on the boundaries then it's all right.

Proof

Suppose there exists to solutions ϕ_1 and ϕ_2 of the given electrostatic problem satisfying Poisson's equation $\boldsymbol{\nabla^2 \phi = -\frac{\rho}{\epsilon_0}}$ and the given boundary condition within a volume V bounded by the surface S.

Physical experience leads us two types of boundary conditions yielding unique solutions. Either the potential ϕ is specified on the boundary (Dirichlet condition) or, normal derivative of ϕ is specified on the boundary (Neumann condition.). let $\phi = \phi_1 - \phi_2$.

As $\nabla^2\phi_1 = -\frac{\rho}{\epsilon_0}$, and $\nabla^2\phi_2 = -\frac{\rho}{\epsilon_0}$ therefore, $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ inside V.

and ϕ=0 or, $\frac{\partial \phi}{\partial n} = 0$ on the surface S for Dirichlet or Neumann boundary conditions respectively.

Now applying Gausses divergence theorem to the vector $\phi\,\vec{\nabla}\phi$ we find,

$$\oiint_S \phi\,\vec{\nabla}\phi.\overrightarrow{d\mathbf{S}} = \iiint_V (\vec{\boldsymbol{\nabla}} \cdot \phi\vec{\nabla}\phi)\,dV = \iiint_V (\phi\nabla^2\phi + \vec{\nabla}\,\phi.\vec{\nabla}\phi)\,dV = \iiint_V \left(\phi\nabla^2\phi + \left|\vec{\nabla}\,\phi\right|^2\right) dV = \iiint_V \left(\left|\vec{\nabla}\,\phi\right|^2\right) dV \text{.....(I)}$$

[As $\nabla^2\phi = 0$]

The integration on the left-hand side of equation (I) vanishes for both types of boundary conditions.

Therefore $\iiint_V \left(\left|\vec{\nabla}\,\phi\right|^2\right) dV = 0$

Since the integrand is positive definite quantity it must be zero at every point in V for the integral to vanish.

Thus, $\vec{\nabla}\,\phi = 0$. or, $\phi = \text{constant}$ or, $\phi_1 - \phi_2$ = constant inside volume V.

For Dirichlet boundary condition $\phi = 0$. on the surface S, so we have $\phi_1 = \phi_2$ throughout

That means the solution is unique.

For Neumann boundary conditions $\vec{\nabla}\,\phi.\hat{n} = \vec{\nabla}(\phi_1 - \phi_2).\hat{n} = 0$ on surface S.

$\phi_1 - \phi_2$ = constant on surface S.

Since the constant is arbitrary we can take it to be zero and hence solution is again unique that means $\phi_1 = \phi_2$

State and explain Earnshaw's theorem. What is the importance of it?

Free movable charge cannot exist in **stable equilibrium** in **free space** under the influence of electrostatic force **alone.** This is Earnshaw's theorem.

Proof - In free space Laplace's equation $\nabla^2\phi = 0$ $\quad or, \boldsymbol{\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0}$

Or multiplying by q we get, $\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} = 0 \ldots\ldots\ldots\ldots(i)$ [where $w = q\phi$]

For stable equilibrium, $\frac{d^2w}{dx^2}, \frac{d^2w}{dy^2}, \frac{d^2w}{dz^2}$ must each positive so that w is minimum.

Then $\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} = 0$ condition is not valid.

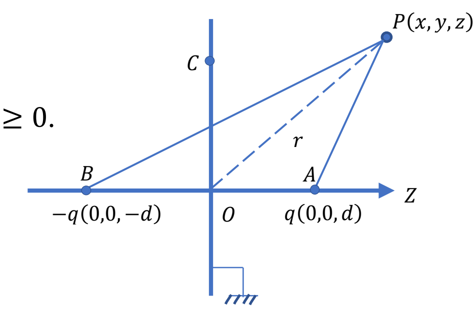
$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$ equation suggest that the condition (i) will satisfied if we consider a negative charge but here the condition of free space is violated.

<u>What is an image charge? And what is the method of electrical image?</u>

For problems outside the conductor the induced charges on the conductor are placed by one or more fictitious point charges on suitable magnitude at suitable location such that the equipotential conditions at the conductor surface and the Laplace's or Poisson's equation is satisfied everywhere in the region outside the conductor. These fictitious, unreal, imaginary charges are known as image charges.

The method useful for finding potential outside a conductor consisting a point charge near it by considering image charges is known as 'Method of Electrical Image'.

<u>A point charge is placed in front of a grounded conducting infinite plane. Use method of images find:</u> a)<u>Potential and field at any point.</u> b) <u>Surface density of induced charge.</u> c) <u>Total charge induced.</u> d) <u>Force between plane and the charge.</u> e) <u>Also calculate the force between the point charge and the induced charge on the plane obtained by method of image by direct integration.</u>



Suppose we have a large grounded conducting plane forming XY-plane. A point charge 'q' is placed at $z = d$. We wise to find the potential and field in the region $z \geq 0$. Symmetry of the problem suggest that an image charge '$-q$' located at $z = -d$ does not affect the conditions,

(i) Potential $\phi = 0$ at all points over the conducting plane.

(ii) ϕ must be zero and infinite.

(iii) ϕ satisfy Laplace's equation in the region $z > 0$, except the point A where charge 'q' is located.

Note that: Condition (i) must be satisfied as any point on the conductor is equidistance from the charge and image charge. Condition (ii) must be satisfied as image charge is considered at a finite distance. Condition (iii) must be satisfied as image charge is considered in the region $z > 0$.

- **<u>Potential at P:</u>** $\boldsymbol{\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{AP} - \frac{q}{BP}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]}$

- **<u>Electric field at P:</u>** $\boldsymbol{\vec{E} = -\vec{\nabla}\phi = -\left[\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right]}$**.**

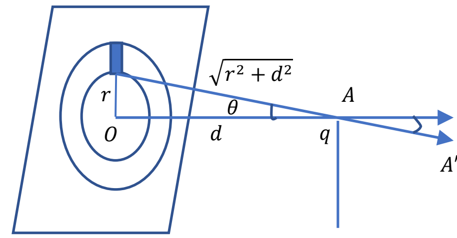
As $\frac{\partial\phi}{\partial x} = \frac{q}{4\pi\epsilon_0}\left[\frac{-x}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{x}{\sqrt{x^2+y^2+(z+d)^2}}\right]$ Similarly we get $\frac{\partial\phi}{\partial y}$ & $\frac{\partial\phi}{\partial z}$.

Hence $$\vec{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{x}{(x^2+y^2+(z+d)^2)^{\frac{3}{2}}} - \frac{x}{(x^2+y^2+(z-d)^2)^{\frac{3}{2}}}\right\}\hat{\imath} + \left\{\frac{y}{(x^2+y^2+(z+d)^2)^{\frac{3}{2}}} - \frac{y}{(x^2+y^2+(z-d)^2)^{\frac{3}{2}}}\right\}\hat{\jmath} + \left\{\frac{(z+d)}{(x^2+y^2+(z+d)^2)^{\frac{3}{2}}} - \frac{(z-d)}{(x^2+y^2+(z-d)^2)^{\frac{3}{2}}}\right\}\hat{k}$$

- **<u>Induced charge density:</u>** If σ be the induced charge surface density then,

$$\sigma = -\epsilon_0\frac{\partial\phi}{\partial z}\Big|_{z=0} = -\epsilon_0\frac{q}{4\pi\epsilon_0}\left[\frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}} + \frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}\right] = -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}$$

- **Total charge induced:** Consider a ring of radii between r and $r+dr$.



Then total charge induced is $Q = \int_0^\infty 2\pi r dr \sigma(r)$.

Then total charge induced is $Q = 2\pi \int_0^\infty -\frac{q}{2\pi}\frac{d}{(r^2+d^2)^{\frac{3}{2}}} rdr = -qd\int_0^\infty \frac{rdr}{(r^2+d^2)^{\frac{3}{2}}}$

$\therefore Q = -qd\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^\infty = -qd\left[0+\frac{1}{d}\right] = -q$

- **Force on charge q by direct integration:** Consider elementary area $rdrd\theta$. Charge on area= $\sigma rdrd\theta$.

Force on q due to elementary area $= \frac{1}{4\pi\epsilon_0}\frac{(d\theta rdr\sigma)q}{(r^2+d^2)} = \frac{q}{4\pi\epsilon_0}\frac{rdr(-q)d}{(r^2+d^2)2\pi(r^2+d^2)^{\frac{3}{2}}}$ [Along $\overrightarrow{AA'}$]

Force on q due to elementary area$= \frac{-q}{4\pi\epsilon_0}\frac{q}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{rdr}{(r^2+d^2)^{\frac{5}{2}}} d\cos\theta = \frac{-q}{4\pi\epsilon_0}\frac{q}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{rdr}{(r^2+d^2)^{\frac{5}{2}}} d\left(\frac{d}{(r^2+d^2)^{\frac{1}{2}}}\right)$, [Along $\overrightarrow{OA}$]

$$\therefore \vec{F} = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{2\pi}2\pi\int_0^\infty \frac{d^2rdr}{(r^2+d^2)^3}, = -\boldsymbol{\frac{q^2}{4\pi\epsilon_0}d^2\int_0^\infty\frac{rdr}{(r^2+d^2)^3}} \text{ [Along } \overrightarrow{OA}]$$

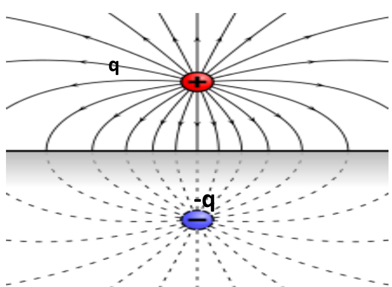
Let $r^2+d^2 = z$ then and $rdr = \frac{dz}{2}$

r	0	∞
z	d^2	∞

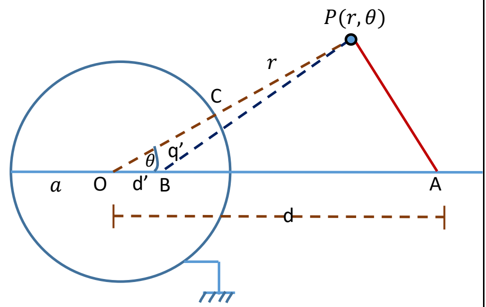
$$\Rightarrow \vec{F} = -\frac{q^2}{4\pi\epsilon_0}d^2\int_{d^2}^\infty\frac{dz}{2z^3} = -\frac{q^2}{4\pi\epsilon_0}\frac{d^2}{2}\left[\frac{z^{-2}}{-2}\right]_{d^2}^\infty = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{4}\frac{1}{d^2} = -\boldsymbol{\frac{q^2}{4\pi\epsilon_0(2d)^2}}\text{, [Along } \overrightarrow{OA}]$$

Note that force on q due to image charge –q is also $-\frac{q^2}{4\pi\epsilon_0(2d)^2}$,[Along $\overrightarrow{OA}$].

- **Line of forces:**



A point charge and a grounded conducting sphere.



Consider a point charge 'q' at a distance 'd' from the center of a grounded conducting sphere of radius 'a'. The symmetry of the problem suggest that the image charge must be placed symmetrically about the line of symmetry OA. Let us try with a single image charge q' located within the sphere at some point B (with OB=d') on the line OA. Now the following conditions-

(i) Potential is zero all over the surface of the sphere.

(ii) Potential ϕ must be zero at infinity.

(iii) Laplace's equation must be satisfied at all outside points excepting at A.

The choice of image charge q' inside the sphere does not affect the condition (i) & (iii). Now the potential ϕ at any point $C(a,\theta)$ on the surface must be zero. Then,

$$\frac{1}{4\pi\epsilon_0}\left[\frac{q}{AC}+\frac{q'}{BC}\right]=0 \Rightarrow \left[\frac{q}{\sqrt{a^2+d^2-2ad\cos\theta}}+\frac{q'}{\sqrt{a^2+d'^2-2ad'\cos\theta}}\right]=0 \quad \therefore \frac{\frac{q}{a}}{\sqrt{1+\frac{d^2}{a^2}-2\frac{d}{a}\cos\theta}}=\frac{-\frac{q'}{d'}}{\sqrt{1+\frac{a^2}{d'^2}-2\frac{a}{d'}\cos\theta}}$$

To make this equality valid for all values of θ we may choose, $\frac{q}{a}=-\frac{q'}{d'}$ and $\frac{d}{a}=\frac{a}{d'}$ $\therefore q'=-\frac{qa}{d}$ as, $d'=\frac{a^2}{d}$

Hence a single image charge $q'=-\frac{qa}{d}$ at a distance $d'=\frac{a^2}{d}$ from center of the sphere and on line OA satisfy all the boundary conditions for our original problem. So image charge $-\frac{qa}{d}$ can now be used instead of induced charge (According to uniqueness theorem).

- **Potential and field at any point $P(r,\theta)$:**

$$\phi(r,\theta)=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{AP}+\frac{q'}{BP}\right]=\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}}+\frac{q'}{\sqrt{r^2+d'^2-2rd'\cos\theta}}\right]$$

and $\vec{E}=-\vec{\nabla}\phi=-\left[\hat{r}\frac{\partial\phi}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right]=E_r\hat{r}+E_\theta\hat{\theta}\ (say)$

$$\therefore E_r=-\frac{\partial\phi}{\partial r}=\frac{1}{4\pi\epsilon_0}\left[\frac{q(r-d\cos\theta)}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}}+\frac{q'(r-d'\cos\theta)}{\left(r^2+d'^2-2rd'\cos\theta\right)^{\frac{3}{2}}}\right] \ \&$$

$$E_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}=\frac{1}{4\pi\epsilon_0}\left[\frac{qd\sin\theta}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}}+\frac{q'd'\sin\theta}{\left(r^2+d'^2-2rd'\cos\theta\right)^{\frac{3}{2}}}\right]$$

- **Introduced charge density:** The surface density of a charge introduced at $C(a,\theta)$

$$\sigma(\theta)=-\epsilon_0\left[\frac{d\phi}{dr}\right]_{r=a}=-\frac{1}{4\pi}\left[\frac{q(a-d\cos\theta)}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}}-\frac{\frac{qa}{d}\left(a-\frac{a^2}{d}\cos\theta\right)}{\left(a^2+\frac{a^4}{d^2}-2\cdot a\cdot\frac{a^2}{d}\cos\theta\right)^{\frac{3}{2}}}\right]$$

$$=-\frac{1}{4\pi}\left[\frac{q(a-d\cos\theta)}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}}-\frac{\frac{qa}{d}\left(a-\frac{a^2}{d}\cos\theta\right)}{\left(\frac{a^2}{d^2}\right)^{\frac{3}{2}}(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}}\right]$$

$$=-\frac{1}{4\pi}\left[\frac{q(a-d\cos\theta)}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}}-\frac{q\frac{a^2}{d^2}\left(a-\frac{a^2}{d}\cos\theta\right)}{\left(\frac{a^2}{d^2}\right)^{\frac{3}{2}}(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}}\right]$$

$$=-\frac{1}{4\pi}\left[\frac{qa-qd\cos\theta-q\frac{d^2}{a}+qd\cos\theta}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}}\right]=-\boldsymbol{\frac{q}{4\pi}\frac{(d^2-a^2)}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}}}$$

Note that, $(a^2 + d^2 - 2rd\cos\theta)^{\frac{1}{2}}$ is the distance of any point on sphere from that point charge 'q'. Then, charge density at a point is inversely proportional to the distance of the point Q. Again as $d^2 > a^2$ then charge density is negative at any point.

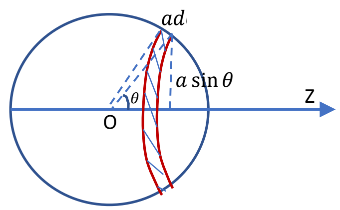
Charge density is maximum at θ=0° and minimum at θ=180°.

$$\sigma_{max} = -\frac{q}{4\pi a}\frac{(d+a)(d-a)}{(d-a)^3} = -\frac{q}{4\pi a}\frac{(d+a)}{(d-a)^2} \;\; and \;\; \sigma_{min} = -\frac{q}{4\pi a}\frac{(d+a)(d-a)}{(d-a)^3} = -\frac{q}{4\pi a}\frac{(d-a)}{(d-a)^2}$$

Note that σ cannot be zero at any point on the sphere.

- **Total induced charge:** Consider a ring of radius $a\sin\theta$ and thickness $ad\theta$.

  Thus total charge (induced) $Q = \int_0^{\pi} \sigma(\theta)\, 2\pi a \sin\theta \cdot ad\theta$



$$\therefore Q = \frac{q}{2}(d^2 - a^2)\int_0^{\pi} \frac{a\sin\theta\, d\theta}{(a^2+d^2-2rd\cos\theta)^{\frac{3}{2}}}$$

Let $(a^2 + d^2 - 2rd\cos\theta) = P^2 \Rightarrow 2ad\sin\theta\, d\theta = 2Pdp$

θ	0	π
P	$d-a$	$d+a$

$$\therefore Q = \frac{q}{2d}(d^2 - a^2)\int_{d-a}^{d+a}\frac{Pdp}{P^3} = \frac{q}{2d}(d^2 - a^2)\left[\frac{1}{P}\right]_{d-a}^{d+a}$$

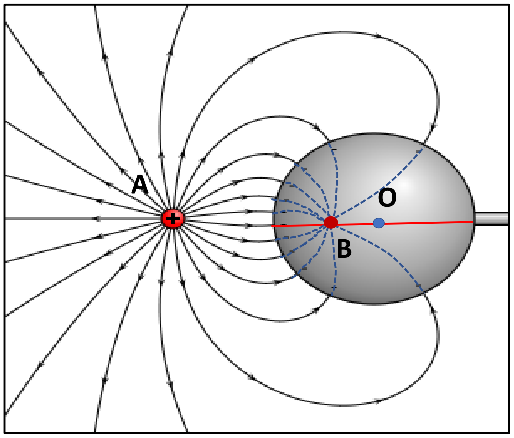
$$= \frac{q}{2d}(d^2 - a^2)\left[\frac{1}{d+a} - \frac{1}{d-a}\right] \Rightarrow \frac{q}{2d}(d^2 - a^2)\left[\frac{d-a-d-a}{(d^2-a^2)}\right] = -\frac{qa}{d}$$

- **Force on** +q:

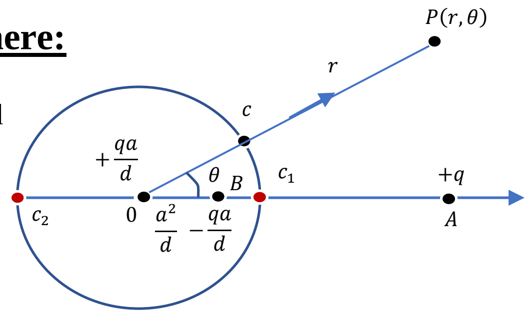
  $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(BA)^2}\hat{k} \Rightarrow \vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{\frac{q^2a}{d}}{\left(d-\frac{a^2}{d}\right)^2}\hat{k} = -\frac{q^2ad}{4\pi\epsilon_0(d^2-a^2)}\hat{k}$. Hence the force is attractive in nature.

  If q is very close to the sphere then, $\vec{F} \rightarrow -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2\delta)^2}\hat{k}$ $[say\ d = a + \delta, \delta\ is\ very\ small]$

- **Lines of forces:**



A point charge and an uncharged insulated conducting sphere:



Suppose a point charge q is placed at a distanced from the center of an insulated conducting sphere of radius a. The following conditions must be satisfied.

(i) ϕ= constant all over the surface of the sphere.

(ii) Net charge on the conductor is zero.

(iii) ϕ satisfy the Laplace's equation at all outside points excepts A.

(iv) ϕ=0 at infinity.

If the sphere is grounded the image charge is $-\frac{qa}{d}$ at a distance $\frac{a^2}{d}$ from center of the sphere.

Now as this image charge and +q charge makes potential of the surface of the sphere 'zero'.

Then an additional image charge (say $+\frac{qa}{d}$) may be placed at '0' to satisfy the condition (i) and (ii). As the image charge is considered inside the sphere the condition (iii) is also satisfied.

- **Potential and field at P(r,θ):** $\phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{\frac{qa}{d}}{\sqrt{r^2+\left(\frac{a^2}{d}\right)^2-2r\frac{a^2}{d}\cos\theta}} + \frac{\frac{qa}{d}}{r}\right]$

Then components of electric field, $E_r = -\frac{\partial\phi}{\partial r} = \frac{1}{4\pi\epsilon_0}\left[\frac{q(r-d\cos\theta)}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qa}{d}\left(r-\frac{a^2}{d}\cos\theta\right)}{\sqrt{r^2+\left(\frac{a^2}{d}\right)^2-2r\frac{a^2}{d}\cos\theta}} + \frac{\frac{qa}{d}}{r^2}\right]$ ……(i)

$$\& \ E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{4\pi\epsilon_0}\left[\frac{qd\sin\theta}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qa}{d}\left(\frac{a^2}{d}\sin\theta\right)}{\left[r^2+\left(\frac{a^2}{d}\right)^2-2r\frac{a^2}{d}\cos\theta\right]^{\frac{3}{2}}}\right]$$

Induced charge density: $\sigma(\theta) = -\epsilon_0\left[\frac{d\phi}{dr}\right]_{r=a} = -\frac{q}{4\pi}\frac{(d^2-a^2)}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}} + \frac{q}{4\pi ad}$ [From eq (i)]

$\therefore \sigma(\theta) = -\frac{q}{4\pi a}\left[\frac{(d^2-a^2)}{(AC)^3} - \frac{1}{d}\right]$ ………(ii)

Then surface density at C_1 i.e. $\sigma_{c_1} = -\frac{q}{4\pi a}\left[\frac{(d^2-a^2)}{(d-a)^3} - \frac{1}{d}\right] = -\frac{q}{4\pi a}\left[\frac{3d-a}{(d-a)^2}\right]$ [Which is negative]

Then surface density at C_2 i.e. $\sigma_{c_2} = -\frac{q}{4\pi a}\left[\frac{(d^2-a^2)}{(d+a)^3} - \frac{1}{d}\right] = +\frac{q}{4\pi a}\left[\frac{3d+a}{(d+a)^2}\right]$ [Which is positive]

Then there must be a **line of no electrification** where surface charge density $\sigma(\theta) = 0$

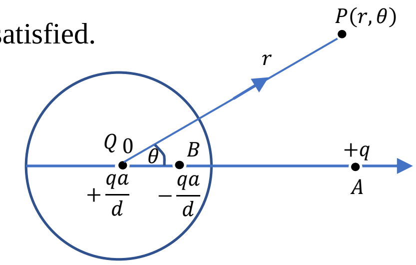
From equation (ii) we get, $(AC)^3 = d(d^2-a^2)$ $\ for\ \sigma(\theta)=0 \Rightarrow (AC)^3 = d^2\left(d-\frac{a^2}{d}\right) = (OA)^2\cdot(OB)$

This represent a circle on the sphere separating positive and negative changes.

- **Force on +q:** $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q\left(-\frac{qa}{d}\right)}{\left(d-\frac{a^2}{d}\right)^2}\hat{k} + \frac{1}{4\pi\epsilon_0}\frac{q\left(\frac{qa}{d}\right)}{d^2}\hat{k}$

A point charge and a charge insulated conducting sphere:

Suppose the conducting sphere carries a charge Q. The following condition must be satisfied.



(i) ϕ= constant all over the surface of the sphere.

(ii) Net charge on the sphere is Q.

(iii) ϕ satisfy the Laplace's equation at all outside points excepts A.

(iv) ϕ=0 at infinity.

To satisfy all the condition our choice is $-\frac{qa}{d}$ at 'B' at a distance $\frac{a^2}{d}$ from 'O' $+\frac{qa}{d}$ at O and additional image charge Q at 'O'.

- **Potential and field at P(r,θ):** $\phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{\frac{qa}{d}}{\sqrt{r^2+\left(\frac{a^2}{d}\right)^2-2r\frac{a^2}{d}\cos\theta}} + \frac{Q+\frac{qa}{d}}{r}\right]$

Then components of electric field, $E_r = -\frac{\partial\phi}{\partial r} = \frac{1}{4\pi\epsilon_0}\left[\frac{q(r-d\cos\theta)}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qa}{d}\left(r-\frac{a^2}{d}\cos\theta\right)}{\sqrt{r^2+\left(\frac{a^2}{d}\right)^2-2r\frac{a^2}{d}\cos\theta}} + \frac{Q+\frac{qa}{d}}{r^2}\right]$ …………(i)

$$\& \ E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{4\pi\epsilon_0}\left[\frac{qd\sin\theta}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qa}{d}\left(\frac{a^2}{d}\sin\theta\right)}{\left[r^2+\left(\frac{a^2}{d}\right)^2-2r\frac{a^2}{d}\cos\theta\right]^{\frac{3}{2}}}\right]$$

Induced charge density: $\sigma(\theta) = -\epsilon_0\left[\frac{d\phi}{dr}\right]_{r=a} = -\frac{q}{4\pi}\frac{(d^2-a^2)}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}} + \frac{Q+\frac{qa}{d}}{4\pi a^2}$ [From eq (i)]

$$\therefore \sigma(\theta) = -\frac{q}{4\pi a}\left[\frac{(d^2-a^2)}{(AC)^3} + \frac{Q+\frac{qa}{d}}{a^2}\right] \text{………(ii)}$$

Then surface density at C_1 i.e. $\sigma_{c_1} = -\frac{q}{4\pi}\frac{3d-a}{(d-a)^2} + \frac{Q}{4\pi a^2}$ [Which is depends on the magnitude of Q]

Then surface density at C_2 i.e. $\sigma_{c_2} = \frac{q}{4\pi d}\frac{3d+a}{(d+a)^2} + \frac{Q}{4\pi a^2}$ [Which is positive]

- **Force on** **+q:** $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q\left(-\frac{qa}{d}\right)}{\left(d-\frac{a^2}{d}\right)^2}\hat{k} + \frac{1}{4\pi\epsilon_0}\frac{q\left(Q+\frac{qa}{d}\right)}{d^2}\hat{k}$

The first term is negative and second term is positive, hence force may be repulsive.

The condition for $\vec{F} = 0$ at $d = d_0$ is $\frac{\frac{qa}{d}}{\left(d_0-\frac{a^2}{d_0}\right)^2} = \frac{Q+\frac{qa}{d}}{{d_0}^2}$ **[where $d < d_0$ force is attractive and when $d > d_0$ force is repulsive]**

In case $Q >> q$, the point of zero is very close to the sphere and is at $d_0 \approx a\left(1+\frac{1}{2}\sqrt{\frac{q}{a}}\right)$.

Suppose the point A is very close to sphere let $d = a+\delta, \delta \ll a$.

Then $\vec{F} \approx \frac{1}{4\pi\epsilon_0}\left[\frac{-qad}{(2ad)^2} + \frac{\left(Q+\frac{qa}{d}\right)}{d^2}\right]\hat{k} \approx \frac{q}{4\pi\epsilon_0}\left[-\frac{q}{4\delta^2} + \frac{Q+q}{d^2}\right]\hat{k}$ [Where $\frac{d}{a} \approx 1$]

If Q be order of q then the force is attractive near sphere.

e) Conductors: electric field and charge density inside and on the surface of a conductor. Force per unit area on the surface. Capacitance of a conductor. Capacitance an isolated spherical conductor. Parallel plate condenser.

1. Explain the following facts:

a) Electric field inside a conductor is zero.

In a perfect conductor there is no resistance to the movement of charges with it. So if there were any field inside a conductor the free charges would move and no static equilibrium would be possible. So we may conclude that the electric field inside a conductor is always zero.

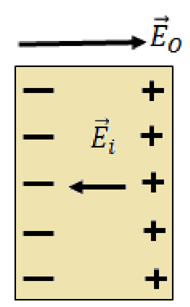
b) Charge given to a conductor reside on its outer surface.

If charge density ρ is nonzero at any point inside a conductor then Gauss's law $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}$ indicates that an electric field $\vec{E}$ exists at that point. As the charge are free to move they move and ultimately take up positions such that electric field inside becomes zero. Now the surface is the only place where the charges can go. They cannot leave the surface because of the existence of surface potential energy barrier.

c) What happen when a conductor is placed in an electric field?

Let a conductor is placed in an electric field it causes the electrons to move opposite with the field and thus produce induced positive & negative charges this movement continuous until the internal field ($\vec{E_i}$) produced charges cancel the external field & the total field inside becomes zero. (Fig 1)

Fig-1



d) "Conductor forms an equipotential region"- Explain.

Any net charge on a conductor lies on its surface and they are distributed in such a way that the electric field $\vec{E}$ inside becomes zero. Then integrating the relation $\vec{E}=-\vec{\nabla}\emptyset=0$, we find that $\emptyset=const.$, i.e. potential $\emptyset$ is constant throughout. The conductor surface forms an equipotential surface otherwise charges would flow from regions of higher potential to those of lower potential.

e) The electric field outside of a conductor is normal to the surface.

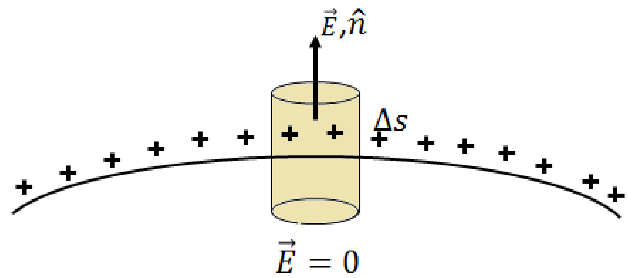
If the electric field is not normal to the surface there will be tangential component and electrons will move around the surface to destroy this tangential component. So in the static equilibrium condition the electric field must be normal to conductor surface. In other words, conductor surface is an equipotential surface $\emptyset=const.$ As $\vec{\nabla}\emptyset$ is a vector normal to the surface $\emptyset=const.$ So we can say that $\vec{E}=-\vec{\nabla}\emptyset$ is a vector normal to the conductor surface.

f) Charge distribution on the surface of a conductor may not be uniform.

Conductor surface forms an equipotential surface $\emptyset=const.$ Poisson's equation $\nabla^2\emptyset=-\frac{\rho}{\epsilon_0}$ indicates that the charge density would depend on the curvature of the surface. The surface density of charge on a conductor surface is greater at point having more curvature.

2. **<u>Show that the electric field just outside a conductor has a magnitude $\frac{\sigma}{\epsilon_0}$ and is directed normal to the conductor surface, where σ is the local surface charge density.</u>**

Let us construct a Gaussian surface in the form of small cylinder box half inside and half outside, with its flat surface parallel and curved surface perpendicular to the conductor surface. As $\vec{E}=0$ inside the conductor and $\vec{E}$ is perpendicular to the conductor surface the electric flux through curved surface and lower flat surface would be zero. So there is contribution to the total flux only from the flat surface outside the conductor. Now from Gauss's law, we have $\vec{E}\cdot\hat{n}\Delta s=\frac{1}{\epsilon_0}\sigma\Delta s$



$$or, E\Delta s=\frac{1}{\epsilon_0}\sigma\nabla s$$

$$or, E=\frac{\sigma}{\epsilon_0}$$

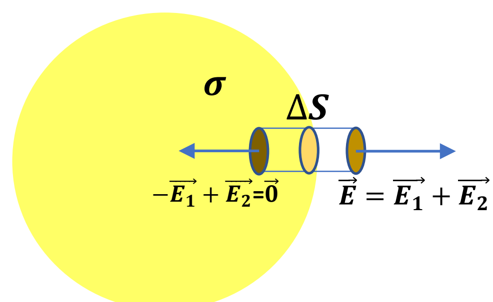
$$\therefore \vec{E}=\frac{\sigma}{\epsilon_0}\hat{n}$$

3. **<u>Show that the mechanical pressure on the surface of a charged conductor is given by $\frac{\sigma^2}{2\epsilon_0}$, where σ is the local surface charge density.</u>**

Electric field at any external point P just outside a charged conductor is given by, $\vec{E}=\frac{\sigma}{\epsilon_0}\hat{n}$ ……….(i)

[Where σ= local surface charge density, $\hat{n}$= outward unit vector normal to the surface.]

This field may be considering as the sum of two fields:



$\vec{E}=\vec{E}_1+\vec{E}_2$……………….(ii) [Where $\vec{E}_1$ is due to charges on a very small area Δs around P and $\vec{E}_2$ is due to rest of the charges.]

For an internal point P' just an opposite of ∇s the contribution $\vec{E}_1$ changes sign such that total field becomes zero, i.e. $0=-\vec{E}_1+\vec{E}_2$…………..(iii)

Solving eq (ii) & (iii) we get, $\vec{E}_1=\vec{E}_2=\frac{\vec{E}}{2}=\frac{\sigma}{2\epsilon_0}\hat{n}$

The charges $\sigma\Delta s$ on Δs cannot a force on itself but this charge $\sigma\Delta s$ exert experiences a force due to field $\vec{E}_2$ produced by other charges. Therefore, force on the charges contained in Δs is, $\vec{F}=\sigma\Delta s\vec{E}_2=\frac{\sigma^2\Delta s}{2\epsilon_0}\hat{n}$.

So the force on unit area or electrostatic pressure is,

$\vec{P}=\frac{\sigma^2}{2\epsilon_0}\hat{n}$ [Obviously the power acts in the outward direction.]

Capacitance

Capacitance is the ratio of the change in an electric charge **in a system** to the corresponding change in its electric potential.

There are two closely related notions of capacitance: ***self-capacitance*** and ***mutual capacitance***.

Any object that can be electrically charged exhibits ***self-capacitance***. A material with a large self-capacitance holds more electric charge at a given voltage than one with low capacitance.

The notion of ***mutual capacitance*** is particularly important for understanding the operations of the capacitor, one of the three elementary linear electronic components (along with resistors and inductors).

The SI unit of capacitance is the Farad (symbol: F), named after the English physicist Michael Faraday. A 1 Farad capacitor, when charged with 1 coulomb of electrical charge, has a potential of 1 volt with respect to reference.

The reciprocal of capacitance is called **elastance.**

Self-capacitance : For an isolated conductor, the amount of electric charge that must be added to an isolated conductor to raise its electric potential w.r.t reference measurement point by one unit (i.e. one volt, in most measurement systems) is called ***self-capacitance***.

The reference point for this potential is a theoretical hollow conducting sphere, of **infinite radius,** with the conductor centered inside this sphere.

That means the idea of 'single conductor' capacitance is a profound one. Here we simply assume that its "counter-charged partner" is situated at a distance of infinity.

Mathematically, the ***self-capacitance*** of a conductor is defined by $C = \frac{Q}{V}$ [**where Q = charge on conductor**

V= potential of conductor w. r. t reference.]

Why earth serves as a reference measurement point for voltages.

We get the capacitance of earth is about 700 microfarads! Hence earth can take huge amounts of charge, **its voltage** or **Potential Difference change w.r.t that imaginary 'partner'** located at infinity distance is **infinitely small,** It serves as a stable, near infinite 'sink' for electrical charge. That is the reason why we find it so useful to use it as a negative terminal or well, the earth. It serves as a reference measurement point for voltages, just the same way as 'mean sea level' does for height measurement on the earth.

Mutual capacitance : When two conductor insulated from each other, usually sandwiching a dielectric material. Then the amount of electric charge that must be added to any conductor to raise its electric potential difference w.r.t another conductor by one unit (i.e. one volt, in most measurement systems) is called ***mutual capacitance***.

Mathematically, the ***mutual capacitance*** of a conductor is defined by $C = \frac{Q}{V}$ [**where Q = charge on conductor**

V= potential difference of two conductors.]

Capacitance of an isolated spherical conductor.

Let we give a charge Q to an isolated spherical conductor. The charge resides on the surface.

The potential on the surface of the spherical conductor is $V = \frac{Q}{4\pi\epsilon_0 R}$

where R is the radius of the conductor.

Then self-capacitance is $C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon_0 R}} = 4\pi\epsilon_0 R$

Parallel plate condenser

Example values of self-capacitance are:

- **for the top "plate" of a van de Graaff generator, typically a sphere 20 cm in radius: 22.24 pF,**
- **the planet Earth: about 710 μF.**