Properties of central force:

A central force is one which at every point at its field is directed either towards or away from some fixed point in the field. It can always be written in the form $\vec{F} = \mp\hat{r}f(\vec{r}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$

[Where $\hat{r}$ is the unit vector in the direction of position vector $\vec{r}$ with the fixed point as origin.]

The force represented by equation (i) is anisotropic force field here the central force depends both direction and magnitude of position vector $\vec{r}$.

Exam: $\vec{F} = \hat{r}\, kr\cos\phi$ .When the central force depends only the magnitude of the position vector then central force is called Centro-symmetric and is expressed as $\vec{F} = \mp\hat{r}f(r)$. Exam: $\vec{F} = \frac{\hat{r}k}{r^2}$. Note that usually term central force means Centro-symmetric central force.

For all Centro-symmetric force, irrespective of the form of the function it can be shown that $\vec{\nabla}\times\vec{F} = \vec{0}$ Then a Centro-symmetric force is always conservative.

Note that conservative forces are not in general central.

Prove that Centro-symmetric central force is always conservative.

⇨ We have Centro-symmetric central force as $\vec{\boldsymbol{F}} = \hat{\boldsymbol{r}}\boldsymbol{f}(\boldsymbol{r})$

then $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\hat{r}f(r) = \vec{\nabla}\times\frac{\vec{r}}{r}f(r) = \vec{\nabla}\times\vec{r}g(r)$ [Where $g(r) = \frac{f(r)}{r}$]

$$= (\vec{\nabla}\times\vec{r})g(r) + \vec{\nabla}g(r)\times\vec{r} = \vec{0} + g(r)\hat{r}\times\vec{r} = \vec{0} \quad \text{(Proved)}$$

Can you find a potential function corresponding the force $\vec{\boldsymbol{F}} = \hat{\boldsymbol{r}}\, \boldsymbol{kr}\cos\boldsymbol{\phi}$

⇨ Here $\vec{\nabla}\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & (r)\hat{\theta} & (r\sin\theta)\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ kr\cos\phi & (r)\,0 & (r\sin\theta)\,0 \end{vmatrix}$

$$= \frac{1}{r^2\sin\theta}(-kr\sin\phi)(r)\hat{\theta} \neq \vec{0}$$

Hence the force is not conserved. So Potential function is not defined here.

Prove that in a central force field angular momentum of the particle w.r.t force center is constant.

⇨ The torque on any parcel about force center under the action of central force (both anisotropic or Centro-symmetric) $\vec{N} = \vec{r} \times \hat{r} f(\vec{r}) = \vec{0}$

Then if $\vec{L}$ is the angular momentum of the particle about force center, then $\frac{d\vec{L}}{dt} = \vec{N} = \vec{0}$. **Then $\vec{L}$ is constant.**

Prove that in a central force field, the motion of the particle is always confined to a plane.

⇨ We have the relation between position, velocity and angular momentum w.r.t force center is,

$\vec{L} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$ **[where m is the mass of the particle.]**

Then by the property of cross product $\vec{r}$ & $\vec{v}$ must lie in the same plane (note that force center is also lie on this plane.) which is perpendicular to$\vec{L}$. Now for central force, $\vec{L}$ is constant vector (i.e, it has fixed magnitude and direction in space). Then $(\vec{r} \times \vec{v})$ **is a constant vector**. Hence $\vec{r}$ & $\vec{v}$ must lie in a **particular plane.**

Thus motion is always confined to a **particular plane (perpendicular to $\vec{L}$) containing the force center**.

Is the motion of a particle under the force field $\vec{F} = -\hat{r}\, kr \cos\phi$ is confined in a plane?

⇨ As $\vec{F} = -\hat{r}\, kr \cos\phi$ can be expressed in the form$-\hat{r}f(\vec{r})$. Then $\vec{F}$is a central force (anisotropic),

Hence obviously the motion is confined in a plane.

Motion under central force:

We can without any loss of generality, choose a two dimensional (planer) co-ordinate system for description of motion of a particle under central force field, as the motion is confined in a plane. For symmetrical reason a plane polar co-ordinate (r,θ) is the best for central force field problems.

In plane polar co-ordinate the acceleration is represented by $\vec{\boldsymbol{a}} = (\ddot{\boldsymbol{r}} - \boldsymbol{r}\dot{\boldsymbol{\theta}}^2)\hat{\boldsymbol{r}} + (\boldsymbol{r}\ddot{\boldsymbol{\theta}} + 2\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}}$.

Then the equation of motion becomes $\boldsymbol{m}(\ddot{\boldsymbol{r}} - \boldsymbol{r}\dot{\boldsymbol{\theta}}^2)\hat{\boldsymbol{r}} + \boldsymbol{m}(\boldsymbol{r}\ddot{\boldsymbol{\theta}} + 2\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{r}}\boldsymbol{f}(\boldsymbol{r})$

Now the problem is reduced to two differential equation of second order $\boldsymbol{m}(\ddot{\boldsymbol{r}} - \boldsymbol{r}\dot{\boldsymbol{\theta}}^2) = \boldsymbol{f}(\boldsymbol{r})$ &

$\boldsymbol{m}(\boldsymbol{r}\ddot{\boldsymbol{\theta}} + 2\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}}) = \mathbf{0}$

[Hence required four initial condition for a particular solution]

Show that the constancy of $|\vec{\boldsymbol{L}}|$ actually reduced the two dimensional problem of central force(centro-symmetric) in one dimensional motion and allows complete solution based on two integral or constant of motion $|\vec{\boldsymbol{L}}|$& $\boldsymbol{E}$ where $\boldsymbol{E}$ is the total energy of the system.

⇨ We have two differential equations $\boldsymbol{m}(\boldsymbol{r}\ddot{\boldsymbol{\theta}} + 2\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}}) = \mathbf{0}$ **............................(i)**

& $\boldsymbol{m}(\ddot{\boldsymbol{r}} - \boldsymbol{r}\dot{\boldsymbol{\theta}}^2) = \boldsymbol{f}(\boldsymbol{r})$**...(ii)**

Multiplying equation (i) by $\boldsymbol{r}$ we get $m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = 0$ or, $\frac{d}{dt}(mr^2\dot{\theta}) = 0$ or, $mr^2\dot{\theta}$ =constant.

That means angular momentum $|\vec{\boldsymbol{L}}|$ = **constant** $= L\ (say) \therefore\ \dot{\boldsymbol{\theta}} = \frac{\boldsymbol{L}}{\boldsymbol{mr^2}}$

Now from equation (ii) $m\left\{\ddot{r} - r\left(\frac{L}{mr^2}\right)^2\right\} = f(r)$ or, $m\left\{\ddot{r} - r\left(\frac{L}{mr^2}\right)^2\right\} = -\frac{dV}{dr}$

[as for conservative force $\boldsymbol{f}(\boldsymbol{r}) = -\frac{\boldsymbol{dV}}{\boldsymbol{dr}}$ where $\boldsymbol{V}$ = potential energy.]

Then $m\ddot{r} - mr\left(\frac{L}{mr^2}\right)^2 = -\frac{dV}{dr}$

or, $m\ddot{r} = -\frac{dV}{dr} + \frac{L^2}{mr^3} = -\frac{d}{dr}\left(V + \frac{L^2}{2mr^2}\right)$

or multiplying both side by $\dot{\boldsymbol{r}}$ we get $m\ddot{r}\dot{r} = -\frac{d}{dr}\left(V + \frac{L^2}{2mr^2}\right)\dot{r}$

Or, $\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{L^2}{2mr^2}\right)$ or, $\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + V + \frac{L^2}{2mr^2}\right) = 0$

or. $\boldsymbol{\frac{1}{2}m\dot{r}^2 + V + \frac{L^2}{2mr^2}}$ **= Constant.** $= E\ (say)$..................................(iii)

Equation (iii) shows that constancy of $|\vec{L}|$ actually reduced the two dimensional problem of central force (Centro-symmetric) in one dimensional motion.

[Note that $E = \frac{1}{2}m\dot{r}^2 + V + \frac{L^2}{2mr^2} = \frac{1}{2}m\dot{r}^2 + V + \frac{m^2r^4\dot{\theta}^2}{2mr^2} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V = K.E. + P.E.$ =Total mechanical energy. Hence equation (iii) allows complete solution based on two integral of motion $|\vec{L}|$ & E].

Show that from differential equation that a particle under central force field (Centro-symmetric) the total energy is constant.

⇨ Sec the above procedure.

For a particle under central force field find the equation which gives the distance r from the force center as an implicit function of t

⇨ We have $\boldsymbol{L = mr^2(t)\dot{\theta}}$ **=constant.........................(i)**

& $E = \boldsymbol{\frac{1}{2}m\dot{r}^2(t) + V + \frac{L^2}{2mr^2(t)}}$ **= constant. ....(ii) [Note that here $r = r(t)$]**

From equation (ii) $\boldsymbol{\dot{r}^2(t) = \frac{2}{m}\left\{E - V - \frac{L^2}{2mr^2(t)}\right\}}$ **or,** $\boldsymbol{\dot{r} = \sqrt{\frac{2}{m}}\left\{E - V - \frac{L^2}{2mr^2(t)}\right\}^{\frac{1}{2}}}$

or, $\boldsymbol{\frac{dr}{dt} = \sqrt{\frac{2}{m}}\left\{E - V - \frac{L^2}{2mr^2(t)}\right\}^{\frac{1}{2}}}$

or, $\boldsymbol{\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}}\left\{E - V - \frac{L^2}{2mr^2(t)}\right\}^{\frac{1}{2}}} = \int_0^t dt}$

$\boldsymbol{\therefore t = \int_0^t dt = \sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{dr}{\left\{E - V - \frac{L^2}{2mr^2(t)}\right\}^{\frac{1}{2}}}}$ **.............................(iii)**

Find the equation of the path of the motion for a particle under central force (Centro-symmetric).

We have $\dot{\theta} = \frac{L}{mr^2(t)}$ or, $\frac{d\theta}{dt} = \frac{L}{mr^2(t)}$ or, $d\theta = \frac{L}{mr^2(t)}dt$

or, $\int_{\theta_0}^{\theta} d\theta = \frac{L}{m}\int_0^t \frac{dt}{r^2(t)} = \frac{L}{m}\sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{dr}{r^2(t)\left\{E-V-\frac{L^2}{2mr^2(t)}\right\}^{\frac{1}{2}}} = L\int_{r_0}^{r} \frac{dr/r^2(t)}{\left\{2m(E-V)-\frac{L^2}{r^2(t)}\right\}^{\frac{1}{2}}}$

$\therefore \theta = \theta_0 + L\int_{r_0}^{r} \frac{dr/r^2(t)}{\left\{2m(E-V)-\frac{L^2}{r^2(t)}\right\}^{\frac{1}{2}}}$ ....(iv).

Equation (iv) gives the equation of the path of the particle.

Note that hence solution of is obtained in terms of four constant $\vec{L}, E, r_0, \theta_0$

- Again note that if $\vec{L} = \vec{0}$ (i.e., $\vec{r}$ & $\vec{v}$ are in the same direction that means direction of $\vec{v}$ goes through force center)

Then from (iv) $\theta = \theta_0$ and from $E = \frac{1}{2}m\dot{r}^2(t) + V + \frac{L^2}{2mr^2(t)}$ we get $E = \frac{1}{2}m\dot{r}^2(t) + V'$. Hence motion is along straight line through force center.

- Note that solving r & θ as a function of time from equation $\therefore t = \sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{dr}{\left\{E-V-\frac{L^2}{2mr^2(t)}\right\}^{\frac{1}{2}}}$

& $\theta = \theta_0 + \frac{L}{m}\int_0^t \frac{dt}{r^2(t)}$ rather difficult. Then in practice it is often easier to find he path of the particle from equation $\theta = \theta_0 + L\int_{r_0}^{r} \frac{dr/r^2(t)}{\left\{2m(E-V)-\frac{L^2}{r^2(t)}\right\}^{\frac{1}{2}}}$ by eliminating t.

Differential equation of the orbit for motion under central force.

We have the equation $m(\ddot{r} - r\dot{\theta}^2) = f(r)$ & $mr^2\dot{\theta} = L$. Then $m\ddot{r} = f(r) + \frac{L^2}{mr^3}$ ... ... (i)

Then $\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{L}{mr^2}$

Then $\ddot{r} = \frac{d}{dt}\frac{dr}{dt} = \frac{d}{dt}\left(\frac{dr}{d\theta}\frac{L}{mr^2}\right) = \frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{L}{mr^2}\right)\frac{d\theta}{dt}$

$$= \left\{\frac{d^2r}{d\theta^2}\frac{L}{mr^2} + \frac{dr}{d\theta}\frac{d}{d\theta}\left(\frac{L}{mr^2}\right)\right\}\dot{\theta} = \left[\frac{d^2r}{d\theta^2}\frac{L}{mr^2} + \frac{dr}{d\theta}\left\{\frac{d}{dr}\left(\frac{L}{mr^2}\right)\frac{dr}{d\theta}\right\}\right]\dot{\theta}$$

$$= \left[\frac{d^2r}{d\theta^2}\frac{L}{mr^2} + \frac{dr}{d\theta}\left\{\left(-2\frac{L}{mr^3}\right)\frac{dr}{d\theta}\right\}\right]\dot{\theta} = \frac{d^2r}{d\theta^2}\frac{L}{mr^2}\frac{L}{mr^2} - 2\frac{L}{mr^3}\left(\frac{dr}{d\theta}\right)^2\frac{L}{mr^2} = \frac{L^2}{m^2r^4}\frac{d^2r}{d\theta^2} - 2\frac{L^2}{m^2r^5}\left(\frac{dr}{d\theta}\right)^2$$

We get from equation (i) $m\frac{L^2}{m^2r^4}\frac{d^2r}{d\theta^2} - 2m\frac{L^2}{m^2r^5}\left(\frac{dr}{d\theta}\right)^2 = f(r) + \frac{L^2}{mr^3}$

then finally, $\frac{d^2r}{d\theta^2}-\frac{2}{r}\left(\frac{dr}{d\theta}\right)^2-r=\frac{mr^4}{L^2}f(r)$ ……………(ii)

- **Note that –Equation (ii) is the differential equation for the orbit, when** $f(r)$ **is known differential equation can be solved [solution is** $r=r(\theta)$**] for the orbit.**
- **Conversely if solution** $r=r(\theta)$ **is known then one can find the force law** $f(r)$

Alternative (but useful) form

We have the equation $m(\ddot{r}-r\dot{\theta}^2)=f(r)$ & $mr^2\dot{\theta}=L$. Then $m\ddot{r}=f(r)+\frac{L^2}{mr^3}$ ………… (*i*)

Let $r=\frac{1}{u}$ then, $\frac{dr}{dt}=\frac{dr}{du}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}=-\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}=-r^2\dot{\theta}\frac{du}{d\theta}=-\frac{L}{m}\frac{du}{d\theta}$

Hence $\ddot{r}=\frac{d}{dt}\frac{dr}{dt}=\frac{d}{dt}\left(-\frac{L}{m}\frac{du}{d\theta}\right)=-\frac{L}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right)=-\frac{L}{m}\frac{d^2u}{d\theta^2}\dot{\theta}=-\frac{L}{m}\frac{d^2u}{d\theta^2}\frac{L}{mr^2}=-\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}$

Then when $r=\frac{1}{u}$ we get from equation (i) $-m\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}=f\left(\frac{1}{u}\right)+\frac{L^2u^3}{m}$.

Hence finally $\frac{d^2u}{d\theta^2}+u=-\frac{m}{L^2u^2}f\left(\frac{1}{u}\right)$ ………….(*ii*)

- **Note that –Equation (ii) is the differential equation for the orbit, when** $f\left(\frac{1}{u}\right)=f(r)$ **is known differential equation can be solved [solution is** $u=u(\theta)$**] for the orbit.**
- **Conversely if solution** $u=u(\theta)$ **is known then one can find the force law** $f\left(\frac{1}{u}\right)=f(r)$

Show that total energy of a particle of mass m **acted on by a central force is given by** $E=\frac{L^2}{2m}\frac{1}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2+r^2\right]+V(r)$ **where symbols are as usual.**

We have $E=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r)$. Now $\dot{r}=\frac{dr}{dt}=\frac{dr}{d\theta}\frac{d\theta}{dt}=\frac{dr}{d\theta}\dot{\theta}=\frac{dr}{d\theta}\frac{L}{mr^2}$ [as $mr^2\dot{\theta}=L$ tnen $\dot{\theta}=\frac{L}{mr^2}$]

Then $E=\frac{1}{2}m\left[\left(\frac{dr}{d\theta}\frac{L}{mr^2}\right)^2+r^2\left(\frac{L}{mr^2}\right)^2\right]+V(r)$ $\therefore E=\frac{L^2}{2m}\frac{1}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2+r^2\right]+V(r)$ (proved)

Show that total energy of a particle of mass m **acted on by a central force is given by** $E=\frac{L^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]+V(\frac{1}{u})$ **where symbols are as usual.**

We have $E=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r)$.

Now $\dot{r}=\frac{dr}{dt}=\frac{d}{dt}\left(\frac{1}{u}\right)==-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{Lu^2}{m}=-\frac{L}{m}\frac{du}{d\theta}$ [As $mr^2\dot{\theta}=L$ $\therefore\dot{\theta}=\frac{L}{mr^2}=\frac{Lu^2}{m}$]

Then $E=\frac{1}{2}m\left[\left(-\frac{L}{m}\frac{du}{d\theta}\right)^2+\frac{1}{u^2}\left(\frac{Lu^2}{m}\right)^2\right]+V\left(\frac{1}{u}\right)$ $\therefore E=\frac{L^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]+V(\frac{1}{u})$ (proved) (alternate form)

Find the equation of orbit for a particle under central force from energy consideration

We have $E = \frac{L^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V(\frac{1}{u})$

Then $\frac{dE}{dr} = \frac{L^2}{2m}\left[\frac{d}{dr}\left(\frac{du}{d\theta}\right)^2 + \frac{d}{dr}(u^2)\right] + \frac{dV(\frac{1}{u})}{dr} = \frac{L^2}{2m}\left[\frac{d}{dx}(x)^2\frac{dx}{dr} + \frac{d}{du}(u^2)\frac{du}{dr}\right] + \frac{dV(\frac{1}{u})}{dr}$ $\quad\left[\frac{du}{d\theta} = x\right]$

$$= \frac{L^2}{2m}\left[2x\frac{dx}{d\theta}\frac{d\theta}{du}\frac{du}{dr} + 2u\frac{du}{dr}\right] + \frac{dV}{dr} = \frac{L^2}{2m}\left[2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\frac{d\theta}{du}(-u^2) + 2u(-u^2)\right] + \frac{dV}{dr} = \frac{-2L^2u^2}{2m}\left[\frac{d^2u}{d\theta^2} + u\right] + \frac{dV}{dr}$$

For central force problem E= constant, $\frac{dE}{dr} = 0$ then $\frac{-L^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -\frac{dV}{dr} = f(r)$

[as for central force $-\frac{dV}{dr} = f(r)$] $\therefore$ $Finaly \frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}f\left(\frac{1}{u}\right)$

Alternate form Let's start with $E = \frac{L^2}{2m}\frac{1}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] + V(r)$

Then $\frac{dE}{dr} = \frac{L^2}{2m}\left(\frac{-4}{r^5}\right)\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] + \frac{L^2}{2m}\frac{1}{r^4}\left[\frac{d}{dr}\left(\frac{dr}{d\theta}\right)^2 + 2r\right] + \frac{dV}{dr}$

Now $\frac{d}{dr}\left(\frac{dr}{d\theta}\right)^2 = \frac{d}{dx}(x)^2\frac{dx}{dr} = \frac{d}{dx}(x)^2\frac{dx}{d\theta}\frac{d\theta}{dr} = 2x\frac{dx}{d\theta}\frac{d\theta}{dr} = 2\frac{dr}{d\theta}\frac{d^2r}{d\theta^2}\frac{d\theta}{dr} = 2\frac{d^2r}{d\theta^2}$ $\quad$ [where $\frac{dr}{d\theta} = x$]

Then $\frac{dE}{dr} = \frac{L^2}{2m}\left(\frac{-4}{r^5}\right)\left(\frac{dr}{d\theta}\right)^2 + \frac{L^2}{2m}\left(\frac{-4}{r^5}\right)r^2 + \frac{L^2}{2m}\frac{1}{r^4}2\frac{d^2r}{d\theta^2} + \frac{L^2}{2m}\frac{1}{r^4}2r + \frac{dV}{dr}$

For central force problem E= constant, $\frac{dE}{dr} = 0$ $\therefore \frac{-2}{m}\frac{L^2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{2}{m}\frac{L^2}{r^3} + \frac{L^2}{mr^4}\frac{d^2r}{d\theta^2} + \frac{L^2}{mr^3} = -\frac{dV}{dr} = f(r)$ [as for central force $-\frac{dV}{dr} = f(r)$]

$\frac{-2}{m}\frac{L^2}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{L^2}{mr^4}\frac{d^2r}{d\theta^2} - \frac{L^2}{mr^3} = f(r)$ Then finally, $\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = \frac{mr^4}{L^2}f(r) \ldots\ldots\ldots\ldots\ldots (i)$

Motion under inverse square law of force.

Here in this case $\overrightarrow{F(r)} = \pm\frac{k}{r^2}\hat{r}$ [we take here k to be positive.] then we can write $-\frac{dV}{dr} = \pm\frac{k}{r^2}$ $\therefore$ $V(r) = \pm\frac{k}{r}$

Note that $E = \frac{1}{2}m\dot{r}^2 + V + \frac{L^2}{2mr^2}$ $\quad then\ E = \frac{1}{2}m\dot{r}^2 \pm \frac{k}{r} + \frac{L^2}{2mr^2} = \frac{1}{2}m\dot{r}^2 + V'$

[where Effective potential $V' = \pm\frac{k}{r} + \frac{L^2}{2mr^2}$]

Show that the path of a particle under central force of inverse square law is in general a conic.

We have the equation $\frac{d^2u}{d\phi^2} + u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right)$..(i)

For central force of inverse square law $F = f(r) = \pm\frac{k}{r^2}$

then $f\left(\frac{1}{u}\right) = \pm ku^2$

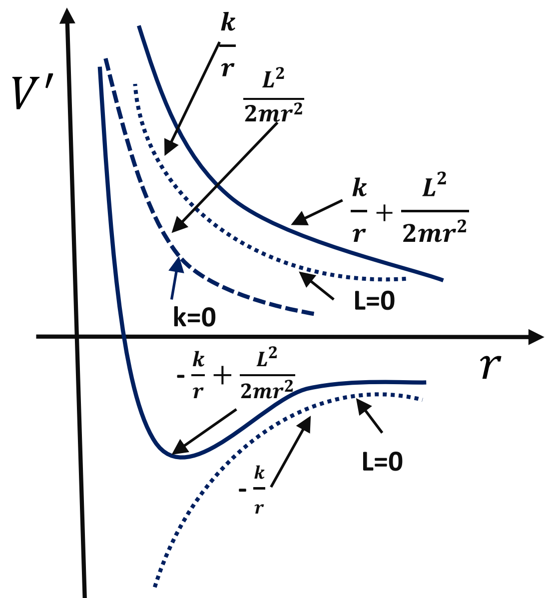
Then from equation (i) we have $\frac{d^2u}{d\phi^2} + u \pm \frac{mk}{L^2} = 0$

Let $\quad u \pm \frac{mk}{L^2} = y \qquad$ then $\frac{d^2y}{d\phi^2} + y = 0$....................................(ii)

The general solution of the equation (ii) $y = A\cos(1.\phi - \phi_0)$

Then finally $\quad u = \mp\frac{mk}{L^2} + A\cos(\phi - \phi_0)$

Or, $\frac{1}{r} = \mp\frac{mk}{L^2} + A\cos(\phi - \phi_0)$ $\qquad$ $\boxed{\therefore \frac{\mp(\frac{L^2}{mk})}{r} = 1 + \left(\frac{A}{\mp\frac{mk}{L^2}}\right)\cos(\phi - \phi_0)}$



Hence orbit equation is in general an equation of conic section in a plane polar co-ordinate with focus at $r = 0$ with semi-latus rectum $l = (\frac{L^2}{mk})$ and eccentricity $\varepsilon = \left(\frac{A}{\mp\frac{mk}{L^2}}\right)$.

[Note that: if we solely interested in the orbit equation then ϕ_0 is irrelevant. because it locates the initial position of the particle on the orbit.]

- **Note that as** $\quad -1 \le \cos(\phi - \phi_0) \le 1 \quad$ then $\frac{1}{r} = \mp\frac{mk}{L^2} + A(\pm 1) = \mp\frac{mk}{L^2} \pm A$.........................(i)

For repulsive force $f(r) = +\frac{k}{r^2}$ then, $\frac{1}{r} = -\frac{mk}{L^2} \pm A$

so turning points are given by $\frac{1}{r_1} = -\frac{mk}{L^2} + A$ and $\frac{1}{r_2} = -\frac{mk}{L^2} - A$

As r cannot be negative then there is only one turning point in case of repulsive force.

For attractive force $f(r) = -\frac{k}{r^2}$ then, $\frac{1}{r} = \frac{mk}{L^2} + A(\pm 1)$

so turning points are given by $\frac{1}{r_1} = \frac{mk}{L^2} + A$ and $\frac{1}{r_2} = \frac{mk}{L^2} - A$

Then for $\frac{mk}{L^2} \ge A$ there are two turning points.

But for $\frac{mk}{L^2} < A$ as r cannot be negative then there is only one turning point.

- Note that, we have $\dot{r} = \sqrt{\frac{2}{m}}\left\{E - V - \frac{L^2}{2mr^2(t)}\right\}^{\frac{1}{2}}$. At turing points $\dot{r} = 0$ then $E \mp \frac{k}{r(t)} - \frac{L^2}{2mr^2(t)} = 0$

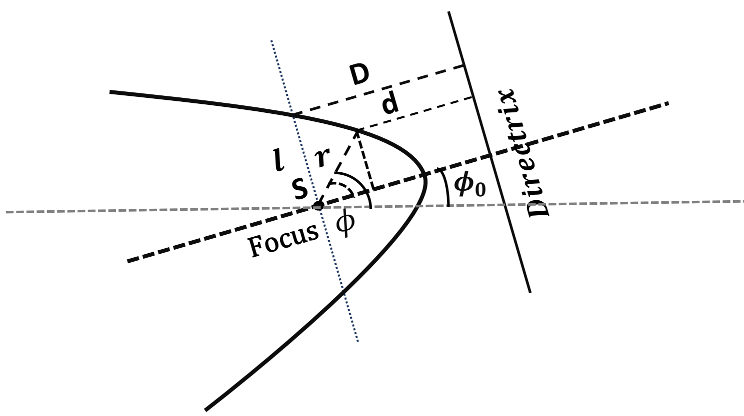
Then $\frac{1}{r}$ is also the solution of the equation. $\frac{L^2}{2mr^2(t)} \pm \frac{k}{r(t)} - E = 0$

Then $\frac{1}{r} = \frac{\mp k \pm \sqrt{k^2 - 4\frac{L^2}{2m}(-E)}}{2\frac{L^2}{2m}} = \mp \frac{mk}{L^2} \pm \frac{\sqrt{k^2 + \frac{2L^2E}{m}}}{\frac{L^2}{m}}$..................(ii)

Comparing equation (i) and (ii) we get that $A = \frac{\sqrt{k^2 + \frac{2L^2E}{m}}}{\frac{L^2}{m}}$

Then eccentricity$\varepsilon = \left(\frac{A}{\mp\frac{mk}{L^2}}\right) = \mp\left(\frac{\sqrt{k^2 + \frac{2L^2E}{m}}}{\frac{mkL^2}{L^2\,m}}\right) = \sqrt{1 + \frac{2EL^2}{mk^2}} = \sqrt{1 + \frac{2El}{k}}$...............(iii)

Equation (ii) relates eccentricity (ε) with semi-latus rectum (l) through energy (E) of the system and force constant (k).



General equation of a conic is $\frac{l}{r} = 1 + \varepsilon\cos(\phi - \phi_0)$ with semi-latus-rectum l and eccentricity ε

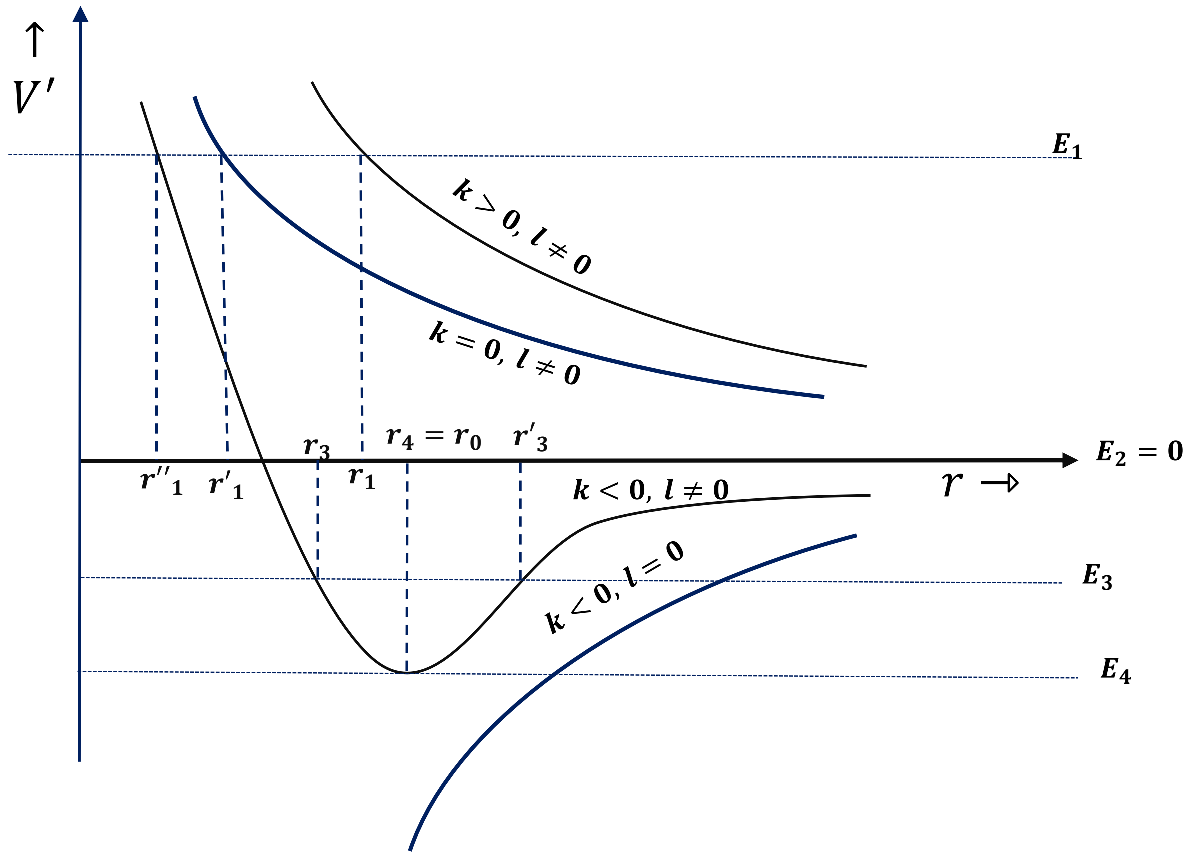
Note that: When $\varepsilon < 1$ the conic is ellipse.

When $\varepsilon = 0$ the conic circle.

When $\varepsilon = 1$ the conic is parabola

When $\varepsilon > 1$ the conic is hyperbola.

Motion with different values of k:



Case 1 k>0 for repulsive force:

we have the total energy as $E = \frac{1}{2}m\dot{r}^2(t) + V + \frac{L^2}{2mr^2(t)}$ then for$k > 0$ the total energy is positive then the particle comes in from $r = \infty$ to a turning point and travels out to infinity again. Motion is thus unbounded and not periodic in r. For example a particle with total energy E_1. The turning point lies at $r = r_1$

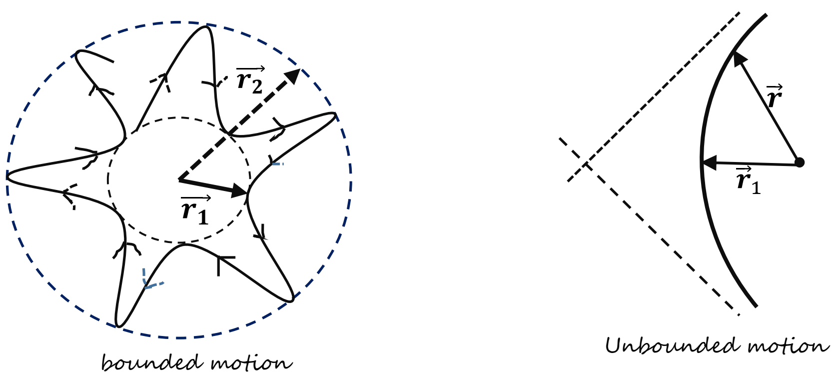
Case 2 k=0 for no force: then turning point occur at smaller values of $r = r'_1 < r_1$

Case 3 k<0 for attractive force:

If total energy is positive then motion is unbounded the turning point occur at $r = r''_1 < r'_1 < r_1$. That means turning point occur at smaller value of r than in the case $k > 0$ and $k = 0$. The particle with this energy can never come closer than r''_1.

But if the energy is negative and greater than minimum potential energy $V'(r) = -\frac{1}{2}\frac{mk^2}{L^2}$ that means energy between 0 & $-\frac{1}{2}\frac{mk^2}{L^2}$ there are two turning point and the motion is bounded. For example a particle with total energy E_3. The turning points lies at $r = r_3$ & r'_3. The lower bound is r_3 and the upper bound is r'_3 the orbits are not necessarily closed but are bounded and contained between two circles of radius r_3 & r'_3. We shall show later on that the orbit is, in fact an ellipse.

__Case 4__ for minimum potential: for energy E_4 in this case two bounds r_3 & r'_3 coincide and the motion is possible only at one radius $r_4 = r_0$ hence orbit is a circle. We will see later that $r_0 = \frac{L^2}{mk}$



Calculate the minimum effective potential energy for attractive force:

we have $V' = -\frac{k}{r} + \frac{L^2}{2mr^2}$ for minimum value of V' let for $r = r_0$ we have $\frac{dV'}{dr} = 0$

then $\frac{k}{{r_0}^2} - \frac{L^2}{m{r_0}^3} = 0$ or, $r_0 = \frac{L^2}{mk}$ Then $V'_{min} = -\frac{k}{\frac{L^2}{mk}} + \frac{L^2}{2m(\frac{L^2}{mk})^2} = -\frac{1}{2}\frac{mk^2}{L^2}$

Stability of orbit and condition for closure:

The condition of stability in radial motion is given by the existence of a local minimum in V' i.e. $\frac{\partial^2 V'}{\partial r^2} > 0$

For any central force $\quad f(r) = \quad kr^n$ potential $V = -kr^{n+1}$

Then $V' = -kr^{n+1} + \frac{L^2}{2mr^2} \quad \therefore \frac{\partial V'}{\partial r} = -k(n+1)r^n - \frac{L^2}{mr^3}$

For maximum or minimum at $r = r_0$ $\frac{\partial V'}{\partial r} = 0$ then $k(n+1){r_0}^n = -\frac{L^2}{m{r_0}^3}$

$\therefore k(n+1) = -\frac{L^2}{m}{r_0}^{-3-n}$

Again $\frac{\partial^2 V\prime}{\partial r^2} = -kn(n+1)r^{n-1} + \frac{3L^2}{mr^4}$

then $\left(\frac{\partial^2 V\prime}{\partial r^2}\right)_{r=r_0} = -kn(n+1){r_0}^{n-1} + \frac{3L^2}{m{r_0}^4}$

$$= -n\left(-\frac{L^2}{m}{r_0}^{-3-n}\right){r_0}^{n-1} + \frac{3L^2}{m{r_0}^4} = \frac{(n+3)L^2}{m{r_0}^4}$$

Then $\left(\frac{\partial^2 V\prime}{\partial r^2}\right)_{r=r_0} > 0$ for $(n+3) > 0$ or, $n > -3$

Therefore for any circular orbit with $\boldsymbol{r = r_0}$ under any central force can satisfy the stability condition if $\boldsymbol{n > -3}$

Condition for closure: an orbit is said to be closed if the particle eventually retrace the path condition are stated as follows:

(i) All bound orbits are closed only for inverse square law of force of Colombian OR, Newton type and for linear law of force Hooke’s type.

(ii) The condition for bound motion is that there is a bounded domain of r in which $V’ \leq E$. The condition for stability of the orbit is $\boldsymbol{n > -3}$ **where** $f(r) = kr^n$.the closed exists only for $n = -1$ and $n = -2$.